ANSWERS

FOR THE
IB DIPLOMA
PROGRAMME

Mathematics

APPLICATIONS AND INTERPRETATION HL

EXAM PRACTICE WORKBOOK

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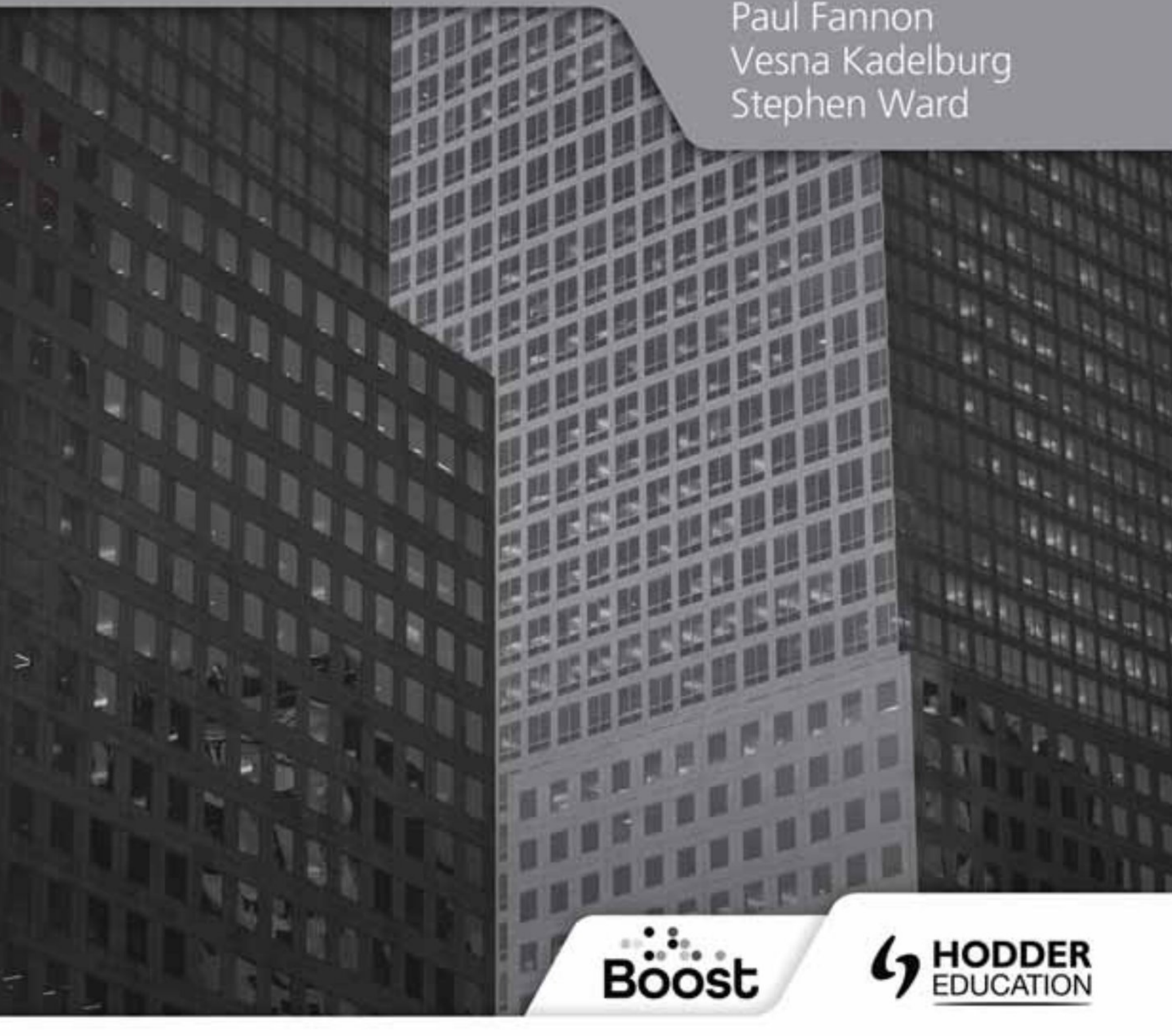


Boost
HODDER EDUCATION

Answers to Practice Questions

1 Number and algebra

1 Just plug the numbers into your calculator. The answer is 4×10^{80}

2 $3 \times 10^{n+1} - 4 \times 10^n = 10^n(3 \times 10^1 - 4)$
$= 10^n(30 - 4)$
$= 26 \times 10^n$
$= 2.6 \times 10 \times 10^n$
$= 2.6 \times 10^{n+1}$

3 We can write this expression as

$$\left(\frac{6}{8}\right) \times \left(\frac{10^n}{10^{-n}}\right) = 0.75 \times 10^{n-(-n)}$$
$$= 7.5 \times 10^{-1} \times 10^{2n}$$
$$= 7.5 \times 10^{2n-1}$$

4 The first term is 20. The common difference is -3. Therefore

$$u_{25} = 20 + (-3) \times (25 - 1)$$
$$= 20 - 3 \times 24$$
$$= -52$$

Tip: You could have just used $u_1 = 20, u_{n+1} = u_n - 3$ in your calculator sequence function.

5 $u_n = 1602 = 21 + 17(n - 1)$

$1581 = 17(n - 1)$

$93 = n - 1$

$94 = n$

6 $u_4 = 10 = u_1 + 3d$

$u_{10} = 34 = u_1 + 9d$

Using the calculator to solve these equations simultaneous

$u_1 = -2$

$d = 4$

Therefore

$$u_{20} = -2 + 19 \times 4$$
$$= 74$$

7 $u_1 = 13, d = -3$ so

$$S_{30} = \frac{30}{2}\big(2 \times 13 - 3 \times (30 - 1)\big) = -915$$

8 $S_{20} = \frac{20}{2}(4 + 130) = 1340$

9 The easiest way to deal with sigma notation is to write out the first few terms by substituting in $r = 1, r = 2, r = 3$, etc.:

$S_n = 16 + 21 + 26 \ldots$

So the first term is 16 and the common difference is 5.

10 Your GDC should have a sum function, which can be used for this. The answer will be 26 350, but you should still write down the first term and common difference found in question 9 as part of your working.

11 This is an arithmetic sequence with first term 500 and common difference 100. The question is asking for S_{28}.

$$S_{28} = \frac{28}{2}(2 \times 500 + 100 \times 27) = 51\,800\text{ m}$$

Tip: The hardest part of this question is realizing that it is looking for the sum of the sequence, rather than just how far Ahmed ran on the 28th day.

12 2.4% of \$300 is \$7.20. This is the common difference. After one year, there is \$307.20 in the account, so this is the 'first term'.

$u_{10} = 307.20 + 9 \times 7.20 = \$372$

Tip: The hardest part of this question is being careful with what 'after 10 years' means – it is very easy to be out by one year.

13 a The differences are 1.1, 0.8 and 0.8. Their average is 0.9. When $t = 0.5$ we are looking for the sixth term of the sequence, which would be

$u_6 = 0 + 0.9 \times 5 = 4.5\text{ m s}^{-1}$

b There are many criticisms which could be made about this model – for example:

- There is too little data for it to be reliable.
- There is no theoretical reason given for it being an arithmetic sequence.
- The ball will eventually hit the ground.
- The model predicts that the ball's velocity grows without limit.
- There may be a pattern with smaller differences later on.

14 The first term is 32 and the common ratio is $-\frac{1}{2}$.

$$u_{10} = 32 \times \left(-\frac{1}{2}\right)^9 = -\frac{1}{16}$$

15 The first term is 1 and the common ratio is 2 so

$u_n = 1 \times 2^{n-1}$

If $u_n = 4096$ then $4096 = 2^{n-1}$

There are four ways you should be able to solve this:

- On a non-calculator paper you might be expected to figure out that $4096 = 2^{12}$.
- You can take logs of both side to get $\ln 4096 = (n-1)\ln 2$ and solve for n.
- You can graph $y = 2^{x-1}$ and intersect it with $y = 4096$.
- You can create a table showing the sequence and determine which row 4096 is in.

Whichever way, the answer is 13.

16 $u_3 = u_1 r^2 = 16$

$u_7 = u_1 r^6 = 256$

Dividing the two equations:

$$\frac{u_1 r^6}{u_1 r^2} = \frac{256}{16}$$

$r^4 = 16$

$r = \pm 2$

$u_1 = \frac{16}{r^2} = 4$ for both possible values of r.

17 The first term is 162, the common ratio is $\frac{1}{3}$

$$S_8 = \frac{162\left(1 - \frac{1}{3}^8\right)}{1 - \frac{1}{3}} = \frac{6560}{27} \approx 243$$

Tip: You can always use either sum formula, but generally if r is between 0 and 1 the second formula avoids negative numbers.

18 The easiest way to deal with sigma notation is to write out the first few terms by substituting in $r = 1, r = 2, r = 3$, etc.:

$S_n = 10 + 50 + 250 \ldots$

So the first term is 10 and the common ratio is 5.

19 Your GDC should have a sum function, which can be used for this. The answer will be 24 414 060, but you should still write down the first term and common difference found in question 9 as part of your working.

20 a This is a geometric sequence with first term 50 000 and common ratio 1.2. ‘After 12 days’ corresponds to the 13th term of the sequence so:

$u_{13} = 50\,000 \times 1.2^{12} = 445\,805$

b The model suggests that the number of bacteria can grow without limit.

21 You could use the formula:

$$FV = 2000 \times \left(1 + \frac{4}{100 \times 12}\right)^{12\times10} = 2981.67$$

However, the general expectation is that you would use the TVM package on your calculator for this type of question. Make sure you can get the same answer using your package as different calculators have slightly different syntaxes.

22 Using the TVM package, $i = 5.10\%$

23 Unless stated otherwise, you should assume that the compound interest is paid annually. Using $FV = 200$ and $PV = -100$, the TVM package suggests that 33.35 years are required. Therefore 33 years would be insufficient, so 34 complete years are required.

24 12% annual depreciation is modelled as compound interest with an interest rate of 12%. Using the TVM package the value is \$24 000 to three significant figures.

25 When adjusting for inflation, the ‘real’ interest rate is $3.2 - 2.4 = 0.8\%$. Using this value in the TVM package gives a final value of \$2081.

26 This expression is $2^{-2\times-2} = 2^4 = 16$

27 $(2x)^3 = 2^3 \times x^3 = 8x^3$

28 This is equivalent to $x = \log_{10}(k)$

29 The given statement is equivalent to

$2x - 6 = \ln 5$

So

$2x = \ln 5 + 6$

$x = \frac{1}{2}\ln 5 + 3$

Tip: The answer could be written in several different ways – for example, $\ln(\sqrt{5}\,e^3)$. Generally, any correct and reasonably simplified answer would be acceptable.

30 Using appropriate calculator functions:

$\ln 10 + \log_{10} e \approx 2.30 + 0.434 \approx 2.74$

31 Look at the fourth decimal place. It is six, which is five or more, round up, so the answer is 0.011.

32 The fourth significant figure is zero, so round down to 105 000.

33 $500\,000 \times 0.1235 = 61\,750$; however, it is not appropriate to quote an answer to more than the least accurate original figure, so it should be reported as 60 000.

34 $12.445 \leq x < 12.455$

35 $\left|\frac{45-38}{38}\right| \times 100\% = 18.4\%$

36 The side of the square is between 6.5 and 7.5 cm. Therefore, the area is between 42.25 and 56.25 cm^2. The percentage errors for each of these are:

$\left|\frac{56.25 - 49}{56.25}\right| \times 100\% = 12.9\%$

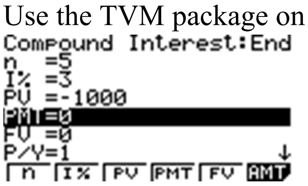
or

$\left|\frac{42.25 - 49}{42.25}\right| \times 100\% = 16.0\%$

so the largest possible error is 16.0%

37 Probabilities cannot be greater than 1.

38 Use the TVM package on your GDC:



Solve for the payment (PMT) variable, getting \$218.35

39 Using TVM with $PV = 100$, PMT = 100, the final value is \$1235.64

Tip: The calculator will give a negative answer – this is because it is treating it as money the bank owes you. You need to be able to interpret the outputs from the TVM package appropriately.

40 Setting $PV = \$50\,000$, $FV = 0$ and $n = 30$, solving for PMT gives a payment of \$2892.

41 Using the simultaneous equation solve on the calculator,

$x = 1, y = 0, z = -1$

42 Using the polynomial equation solve on your GDC:

$x = -0.303, 1$ or 3.30

43 Using the laws of logarithms this becomes $2\log_{10} x + \log_{10} 100 + \log_{10} x$

Which simplifies to $3\log_{10} x + 2$

44 $8^{\frac{2}{3}} = \left(8^{\frac{1}{3}}\right)^2 = \left(\sqrt[3]{8}\right)^2 = 2^2 = 4$

Tip: Obviously in an exam this can be checked on your calculator, but it is useful to be able to do this manually to improve your understanding.

45 $\frac{1}{(4x)^{\frac{1}{2}}} = \frac{1}{\sqrt{4x}} = \frac{1}{2\sqrt{x}}$

46 The first term is 2, the common ratio is $\frac{1}{3}$, so

$$S_\infty = \frac{2}{1 - \frac{1}{3}} = 3$$

47 The common ratio is $2x$. This will converge if $|2x| < 1$ which is when $|x| < \frac{1}{2}$. You could also write this as $-\frac{1}{2} < x < \frac{1}{2}$.

48 $(1 + i)(1 - 2i) = 1 + 2 + i - 2i = 3 - i$

The real part of this number is 3.

49 If $z = a + ib$ then $\text{Im}(z) = b$. Therefore $a + b + ib = 2 - 2i$

Comparing imaginary parts, $b = -2$.

Comparing real parts, $a + b = 2$ therefore $a = 4$.

Therefore $z = 4 - 2i$

50 $4 + i$

51 Let $z = a + ib$

$(a + ib) + 2(a - ib) = 4 + 6i$

$3a - ib = 4 + 6i$

Comparing real and imaginary parts:

$3a = 4$ so $a = \dfrac{4}{3}$

$-b = 6$ so $b = -6$

So, $z = \frac{4}{3} - 6i$

52 $|z| = \sqrt{1^2 + (-\sqrt{3})^2} = \sqrt{4} = 2$

$\tan(\arg z) = \frac{-\sqrt{3}}{1}$

$\arg z = -\frac{\pi}{3}$ or $\frac{2\pi}{3}$

Considering the position of z on the complex plane, $\arg z = -\frac{\pi}{3}$

($\frac{5\pi}{3}$ would also be an acceptable answer)

Tip: You should also be able to do this using your calculator.

53 $a^2 + 2b^2i^2 + 2abi + abi = a^2 - 2b^2 + 3abi$

54 $\frac{1}{a+bi} = \frac{a-bi}{(a+bi)(a-bi)} = \frac{a-bi}{a^2-b^2i^2} = \frac{a}{a^2+b^2} - \frac{b}{a^2+b^2}i$

So, the real part is $\frac{a}{a^2+b^2}$

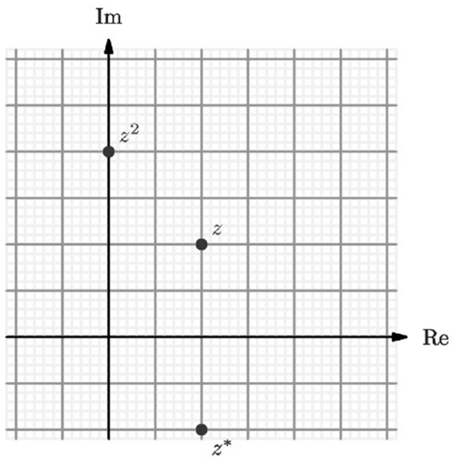
55 $wz = (1-2i)(2+i) = 2 + i - 4i - 2(i^2) = 4 - 3i$

Therefore

$2z + wz = (4+2i) + (4-3i) = 8 - i$

56 16

57



58 $z = \frac{-4 \pm \sqrt{16 - 4 \times 13}}{2}$

$= \frac{-4 \pm \sqrt{-36}}{2}$

$= \frac{-4 \pm 6i}{2}$

$= -2 \pm 3i$

Tip: You should also be able to do this using your calculator.

59 $|z| = 3, \arg z = \frac{\pi}{3}$

60 $|z| = 5, \arg z = \frac{\pi}{4}$

61 $a\cos b + ai\sin b$

62 $|z| = \sqrt{p^2 + p^2} = \sqrt{2}p$

$\arg z = \arctan\left(\frac{p}{p}\right) = \frac{\pi}{4}$

So, $z = \sqrt{2}p \operatorname{cis}\frac{\pi}{4}$

63 $2i$

64 $2\operatorname{cis}\left(-\frac{\pi}{3}\right)$

65 $10\operatorname{cis}(A + B)$

66 $5e^{4i}$

67 $32\operatorname{cis}(5\theta)$

68 $V = \operatorname{Re}\left(3e^{100it} + 4e^{100it+i}\right)$

$= \operatorname{Re}\left(e^{100it}\left(3 + 4e^{i}\right)\right)$

Using a calculator to convert $3 + 4e^{i}$ into polar form

$V = \operatorname{Re}\left(e^{100it} \times 6.16e^{0.578}\right)$

So, $V = 6.16\cos(100t + 0.578)$

69 $7 + 4i$

70 An enlargement factor 2 centred on the origin and a rotation 90 degrees anticlockwise centred on the origin.

71 a 3×2

b 1

72 $\begin{pmatrix} 4 & 2 & 9 \\ 4 & 3 & 4 \end{pmatrix}$

73 $\begin{pmatrix} k & 0 \\ 0 & k^2 \end{pmatrix} - \begin{pmatrix} 2k & 2 \\ 2 & -2k \end{pmatrix} = \begin{pmatrix} -k & -2 \\ -2 & k^2 + 2k \end{pmatrix}$

74 $\begin{pmatrix} 5 & 4 \\ 1 & 0 \end{pmatrix}$

75 $\begin{pmatrix} k \times 1 + 3 \times 4 & k \times (-k) + 3 \times 5 \\ 2 \times 1 + 0 \times 4 & 2 \times (-k) + 0 \times 5 \end{pmatrix} = \begin{pmatrix} k + 12 & -k^2 + 15 \\ 2 & -2k \end{pmatrix}$

76 $\begin{pmatrix} 12 & 15 \\ 10 & 18 \end{pmatrix}\begin{pmatrix} 0.4 \\ 0.6 \end{pmatrix} = \begin{pmatrix} 13.8 \\ 14.8 \end{pmatrix}$

The shop made more profit in the second week (\$14.8 versus \$13.8)

77 $\boldsymbol{AB} = \begin{pmatrix} -1 & 3 \\ 4 & -4 \end{pmatrix}$

$\boldsymbol{BA} = \begin{pmatrix} 1 & 1 \\ 2 & -6 \end{pmatrix}$

$\boldsymbol{AB} \neq \boldsymbol{BA}$

78 $\boldsymbol{A}^2 = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 4 \end{pmatrix}$

$3\boldsymbol{A} = \begin{pmatrix} 3 & 0 \\ 0 & 6 \end{pmatrix}$

$\boldsymbol{A}^2 - 3\boldsymbol{A} = \begin{pmatrix} -2 & 0 \\ 0 & -2 \end{pmatrix}$ Therefore $k = 2$

79 7

80 $\frac{1}{7}\begin{pmatrix} 1 & 2 & 2 \\ 1 & 2 & -5 \\ -2 & 3 & -4 \end{pmatrix}$

81 $k \times 1 - k \times k^2 = k - k^3$

82 $\frac{1}{k-k^3}\begin{pmatrix} 1 & -k^2 \\ -k & k \end{pmatrix}$

Note: The condition that $k > 1$ guarantees that $k \neq 1, -1$ or 0 therefore the determinant is not zero so the inverse exists.

83 $\boldsymbol{A} = \begin{pmatrix} 2 & 2 \\ 5 & -1 \end{pmatrix}$

$\mathbf{x} = \begin{pmatrix} x \\ y \end{pmatrix}$

$\mathbf{b} = \begin{pmatrix} 7 \\ 0 \end{pmatrix}$

84 $\begin{pmatrix} 1 & -3 \\ 2 & 4 \end{pmatrix}^{-1} = \frac{1}{10}\begin{pmatrix} 4 & 3 \\ -2 & 1 \end{pmatrix}$

$\begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{10}\begin{pmatrix} 4 & 3 \\ -2 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ k \end{pmatrix} = \frac{1}{10}\begin{pmatrix} 4+3k \\ -2+k \end{pmatrix}$

85 a $\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} H \\ U \\ N \\ D \end{pmatrix} = \begin{pmatrix} D \\ U \\ H \\ N \end{pmatrix}$

$\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} R \\ E \\ D \\ S \end{pmatrix} = \begin{pmatrix} S \\ E \\ R \\ D \end{pmatrix}$

So, it encodes to DUHNSERD

b $\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix}$

c $\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} H \\ A \\ M \\ T \end{pmatrix} = \begin{pmatrix} M \\ A \\ T \\ H \end{pmatrix}$

$\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} T \\ M \\ E \\ A \end{pmatrix} = \begin{pmatrix} E \\ M \\ A \\ T \end{pmatrix}$

$\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} L \\ C \\ I \\ A \end{pmatrix} = \begin{pmatrix} I \\ C \\ A \\ L \end{pmatrix}$

So, it decodes to MATHEMATICAL

Tip: This is an example of a cipher called the transposition cipher

86 $\begin{pmatrix}2 & 0\\1 & -1\end{pmatrix}\begin{pmatrix}3\\1\end{pmatrix} = \begin{pmatrix}6\\2\end{pmatrix} = 2 \times \begin{pmatrix}3\\1\end{pmatrix}$

Therefore $\begin{pmatrix}3\\1\end{pmatrix}$ is an eigenvector with eigenvalue 2.

87 The characteristic polynomial comes from

$\begin{vmatrix}2-\lambda & 4\\3 & 1-\lambda\end{vmatrix} = 0$

$(2-\lambda)(1-\lambda) - 12 = 0$

$\lambda^2 - 3\lambda - 10 = 0$

Solving this using the GDC:

$\lambda = 5 \text{ or } -2$

88 When $\lambda = 5$:

$\begin{pmatrix}2 & 4\\3 & 1\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix} = 5\begin{pmatrix}x\\y\end{pmatrix}$

So:

$2x + 4y = 5x$

$3x + y = 5y$

These simplify to:

$-3x + 4y = 0$

$3x - 4y = 0$

These are both the same, so we are free to choose one value. Let's choose $x = 4$ then $y = 3$ so an eigenvector is $\begin{pmatrix}4\\3\end{pmatrix}$.

When $\lambda = -2$:

$\begin{pmatrix}2 & 4\\3 & 1\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix} = -2\begin{pmatrix}x\\y\end{pmatrix}$

So:

$2x + 4y = -2x$

$3x + y = -2y$

These simplify to:

$4x + 4y = 0$

$3x + 3y = 0$

These are both the same, so we are free to choose one value. Let's choose $x = 1$ then $y = -1$ so an eigenvector is $\begin{pmatrix}1\\-1\end{pmatrix}$.

89 We can write down $\boldsymbol{D} = \begin{pmatrix}5 & 0\\0 & -2\end{pmatrix}$ then $\boldsymbol{P} = \begin{pmatrix}4 & 1\\3 & -1\end{pmatrix}$.

Using a GDC $\boldsymbol{P}^{-1} = \frac{1}{7}\begin{pmatrix}1 & 1\\3 & -4\end{pmatrix}$

Therefore $\begin{pmatrix}2 & 4\\3 & 1\end{pmatrix} = \frac{1}{7}\begin{pmatrix}4 & 1\\3 & -1\end{pmatrix}\begin{pmatrix}5 & 0\\0 & -2\end{pmatrix}\begin{pmatrix}1 & 1\\3 & -4\end{pmatrix}$

90 $M^n = \begin{pmatrix} 2 & 1 \\ 1 & -2 \end{pmatrix} \begin{pmatrix} 1^n & 0 \\ 0 & 0.5^n \end{pmatrix} \begin{pmatrix} 2 & 1 \\ 1 & -2 \end{pmatrix}^{-1}$

As $n \to \infty$ this expression becomes:

$\begin{pmatrix} 2 & 1 \\ 1 & -2 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ 1 & -2 \end{pmatrix}^{-1}$

So, M^n tends to $\frac{1}{5}\begin{pmatrix} 4 & 2 \\ 2 & 1 \end{pmatrix}$

Tip: You could have used your calculator to investigate a very large power of M to see that it quickly gets close to this value.

91 a The birth rate is effectively adults producing juveniles.

Since $J_{n+1} = 0.2A_n + 0.8J_n$ the 0.2 parameterizes the birth rate.

b The characteristic equation is:

$(0.9 - \lambda)(0.8 - \lambda) - 0.02 = 0$

$\lambda^2 - 1.7\lambda + 0.7 = 0$

Using the GDC, $\lambda = 1$ or 0.7

When $\lambda = 1$:

$\begin{pmatrix} 0.9 & 0.1 \\ 0.2 & 0.8 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = 1 \begin{pmatrix} x \\ y \end{pmatrix}$

So,

$-0.1x + 0.1y = 0$

$0.2x - 0.2y = 0$

Choosing $x = 1$ we get $y = 1$ so $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ is an eigenvector.

When $\lambda = 0.7$:

$\begin{pmatrix} 0.9 & 0.1 \\ 0.2 & 0.8 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = 0.7 \begin{pmatrix} x \\ y \end{pmatrix}$

So,

$0.2x + 0.1y = 0$

$0.2x + 0.1y = 0$

Choosing $x = 1$ we get $y = -2$ so $\begin{pmatrix} 1 \\ -2 \end{pmatrix}$ is an eigenvector.

Therefore, the diagonalization is

$\begin{pmatrix} 0.9 & 0.1 \\ 0.2 & 0.8 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & -2 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0.7 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & -2 \end{pmatrix}^{-1}$

$= \frac{1}{3} \begin{pmatrix} 1 & 1 \\ 1 & -2 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0.7 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ 1 & -1 \end{pmatrix}$

c $\begin{pmatrix}0.9 & 0.1\\0.2 & 0.8\end{pmatrix}^n = \frac{1}{3}\begin{pmatrix}1 & 1\\1 & -2\end{pmatrix}\begin{pmatrix}1^n & 0\\0 & 0.7^n\end{pmatrix}\begin{pmatrix}2 & 1\\1 & -1\end{pmatrix}$

As n gets very large

$$\begin{pmatrix}0.9 & 0.1\\0.2 & 0.8\end{pmatrix}^n \to \frac{1}{3}\begin{pmatrix}1 & 1\\1 & -2\end{pmatrix}\begin{pmatrix}1 & 0\\0 & 0\end{pmatrix}\begin{pmatrix}2 & 1\\1 & -1\end{pmatrix}$$

$$= \frac{1}{3}\begin{pmatrix}2 & 1\\2 & 1\end{pmatrix}$$

The initial condition is $\begin{pmatrix}1000\\500\end{pmatrix}$ so the long term population is

$$\frac{1}{3}\begin{pmatrix}2 & 1\\2 & 1\end{pmatrix}\begin{pmatrix}1000\\500\end{pmatrix} = \begin{pmatrix}833.33\\833.33\end{pmatrix}$$

So, the adult population is expected to be approximately 833.

Tip: You could use your calculator with a large value of n to validate this answer. These types of population models are called Leslie matrices and are very common in Mathematical Biology.

2 Functions

1 Rearrange into the form $y = mx + c$:

$$3x - 4y - 5 = 0$$
$$4y = 3x - 5$$
$$y = \frac{3}{4}x - \frac{5}{4}$$

So, $m = \frac{3}{4}, c = -\frac{5}{4}$

2 Use $y - y_1 = m(x - x_1)$:

$$y + 4 = -3(x - 2)$$
$$y + 4 = -3x + 6$$
$$y = -3x + 2$$

3 Find the gradient using $m = \frac{y_2 - y_1}{x_2 - x_1}$:

$$m = \frac{1+5}{9+3} = \frac{1}{2}$$

Use $y - y_1 = m(x - x_1)$:

$$y - 1 = \frac{1}{2}(x - 9)$$
$$2y - 2 = x - 9$$
$$x - 2y - 7 = 0$$

4 Gradient of parallel line is $m = 2$

$$y - 4 = 2(x - 1)$$

$$y = 2x + 2$$

5 Gradient of perpendicular line is $m = -\frac{1}{-\frac{1}{4}} = 4$

$y - 3 = 4(x + 2)$

$y = 4x + 11$

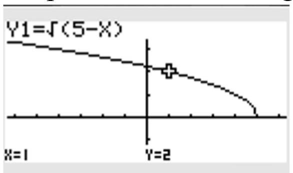
6 Substitute $x = -2$ into the function:

$f(-2) = 3(-2)^2 - 4$
$= 8$

7 $2x - 1 > 0$

$x > \frac{1}{2}$

8 Graph the function using the GDC:

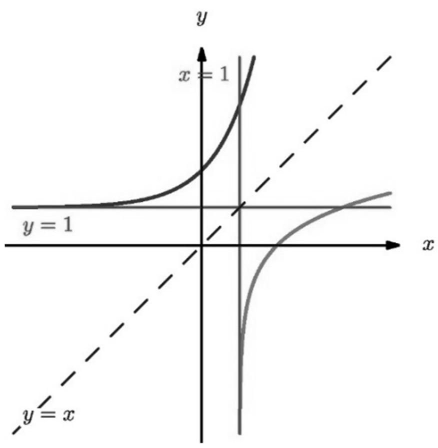


$\mathrm{f}(1) = 2$, so range is $\mathrm{f}(x) \geq 2$

9 To find $\mathrm{f}^{-1}(-8)$, solve $\mathrm{f}(x) = -8$:

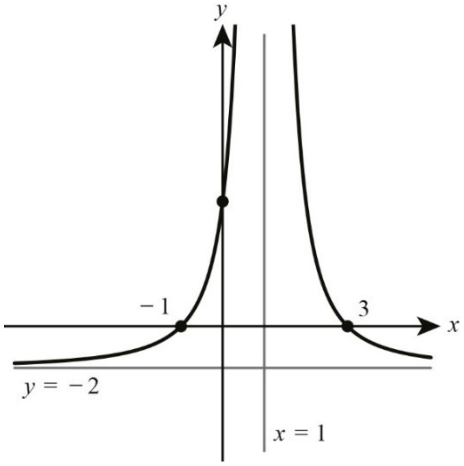
$4 - 3x = -8$
$-3x = -12$
$x = 4$

10 Reflect the graph in the line $y = x$.

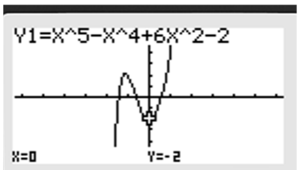


11 Put in the vertical and horizontal asymptote and the x-intercepts (zeros of the function).

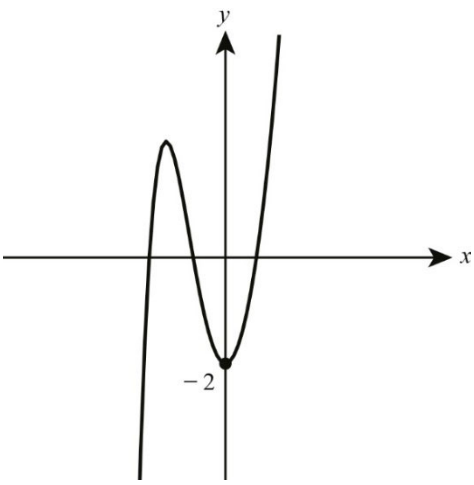
Since $\mathrm{f}(x) > -2$, it must tend to ∞ as it approaches the vertical asymptote from either side.



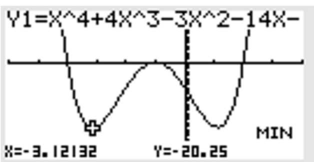
12 Graph the function using the GDC:



The y-intercept is $(0, -2)$. Now sketch the graph from the plot on the GDC:



13 a Graph the function and use 'min' and 'max' to find the coordinates of the vertices, moving the cursor as necessary:

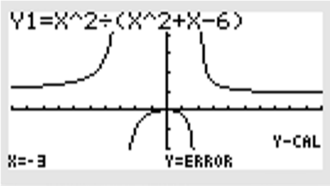


Coordinates of vertices: $(-3.12, -20.3), (-1, 0), (1.12, -20.3)$

b From the graph you can see there is a line of symmetry through the maximum point:

Line of symmetry: $x = -1$

14 Graph the function and look for values where there appear to be asymptotes:



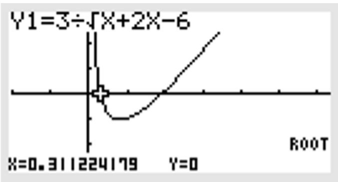
Vertical asymptotes occur at values of x where the y values appears as 'error':

Vertical asymptotes: $x = -3$ and $x = 2$

The y-value approaches 1 as the x-value gets big and positive or big and negative:

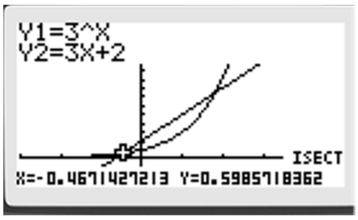
Horizontal asymptote: $y = 1$

15 Graph the function and use 'root', moving the cursor from one to the other:



Zeros: $x = 0.311, 1.92$

16 Graph the function and use 'isct', moving the cursor from one intersection point to the other:



Points of intersection: $(-0.467, 0.599)$ and $(1.83, 7.50)$

17 Substitute each pair of conditions into the model:

$$\begin{cases} m(-5) + c = 10 \\ m(1) + c = -8 \end{cases}$$

$$\begin{cases} -5m + c = 10 \\ m + c = -8 \end{cases}$$

Solve simultaneously using the GDC: $m = -3, c = -5$

So, $\mathrm{f}(x) = -3m - 5$

18 a Substitute $x = 3$ into the function as defined on the domain $0 \le x \le 10$:

$$\begin{aligned} \mathrm{f}(3) &= 0.5(3) + 11 \\ &= 12.5 \end{aligned}$$

b Substitute $x = 11$ into the function as defined on the domain $x > 10$:

$$\begin{aligned} \mathrm{f}(11) &= 2(11) - 4 \\ &= 18 \end{aligned}$$

19 Substitute the three sets of conditions into the model:

$$\begin{cases} a(1)^2 + b(1) + c = -3 \\ a(2)^2 + b(2) + c = 4 \\ a(3)^2 + b(3) + c = 17 \end{cases}$$

$$\begin{cases} a + b + c = -3 \\ 4a + 2b + c = 4 \\ 9a + 3b + c = 17 \end{cases}$$

Solve simultaneously using the GDC: $a = 3, b = -2, c = -4$

So, $\mathrm{f}(x) = 3x^2 - 2x - 4$

20 The x-coordinate of the vertex is $x =-\frac{b}{2a}$:

$$-2 = -\frac{b}{2a} \Rightarrow 4a = b$$

The y-intercept is $x = 0, y = 9$:

$c = 9$

Substitute $x = 1, y = 19$ into $y = ax^2 + bx + 9$:

$19 = a + b + 9$
$a + b = 10$

Since $b = 4a$,

$a + 4a = 10$
$a = 2$

So, $y = 2x^2 + 8x + 9$

21 Substitute each pair of conditions into the model:

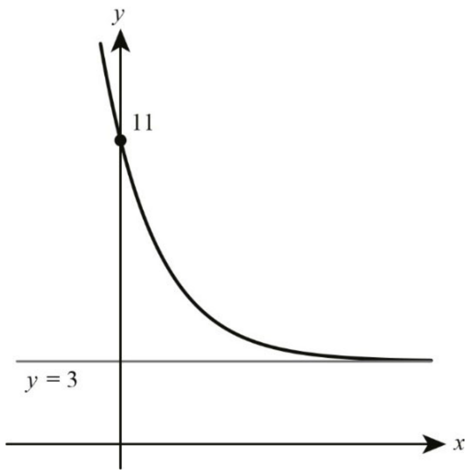
$$\begin{cases} k \times 3^{-(-2)} + c = 38 \\ k \times 3^{-(-1)} + c = 14 \end{cases}$$

$$\begin{cases} 9k + c = 38 \\ 3k + c = 14 \end{cases}$$

Solve simultaneously using the GDC: $k = 4, c = 2$

So, $\mathrm{f}(x) = 4 \times 3^{-x} + 2$

22 Plot on the GDC and identify the horizontal asymptote at $y = 3$:



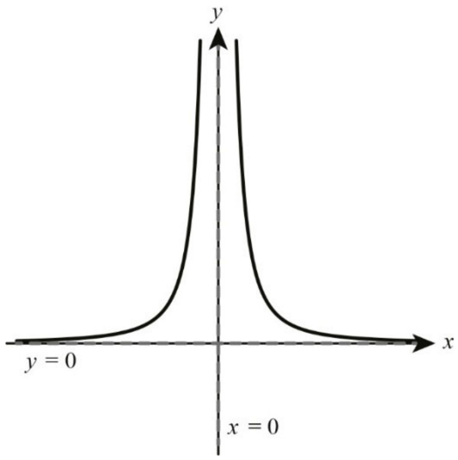
23 Substitute the given condition into $y = \frac{k}{x^2}$:

$$3 = \frac{k}{2^2}$$

$$k = 12$$

So, $y = \frac{12}{x^2}$

24 Plot on the GDC and identify the horizontal and vertical asymptotes:



Horizontal asymptote: $y = 0$

Vertical asymptote: $x = 0$

25 Substitute the four sets of conditions into the model:

$$\begin{cases} a(-2)^3 + b(-2)^2 + c(-2) + d = 1 \\ a(-1)^3 + b(-1)^2 + c(-1) + d = 7 \\ a(1)^3 + b(1)^2 + c(1) + d = 1 \\ a(2)^3 + b(2)^2 + c(2) + d = 13 \end{cases}$$

$$\begin{cases} -8a + 4b - 2c + d = 1 \\ -a + b - c + d = 7 \\ a + b + c + d = 1 \\ 8a + 4b + 2c + d = 13 \end{cases}$$

Solve simultaneously using the GDC: $a = 2, b = 1, c = -5, d = 3$

So, $\mathrm{f}(x) = 2x^3 + x^2 - 5x + 3$

26 The amplitude is a:

$a = 4$

The period is $\frac{360}{b}$:

$$\frac{360}{b} = 240$$

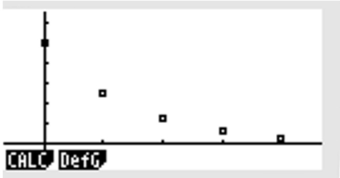
$$b = \frac{3}{2}$$

The principal axis is $y = d$:

$d = -1$

So, $\mathrm{f}(x) = 4\sin\left(\frac{3}{2}x\right) - 1$

27 Plot the data on the GDC to see which model the graph most closely resembles:



The graph seems to have a horizontal asymptote at $y = 0$ but not a vertical asymptote at $x = 0$, so $y = k \times 2^{rx}$ is the most appropriate model.

28 There are two parameters to find (k and r) so form two equations using the first two points:

$$\begin{cases} k \times 2^{r\times 0} = 5 \\ k \times 2^{r\times 1} = 2.5 \end{cases}$$

$$\begin{cases} k = 5 \\ k \times 2^{r} = 2.5 \end{cases}$$

$$5 \times 2^r = 2.5$$
$$2^r = \frac{1}{2}$$
$$r = -1$$

So, $y = 5 \times 2^{-x}$

29 The market share cannot exceed 100%:

$3x + 4 \leq 100$
$x \leq 32$

Also, the amount spent cannot be negative, so a suitable domain is $0 \leq x \leq 32$.

30 The units of x are thousands of \$, so substitute $x = 18$ into the function:

$$s(18) = 3 \times 18 + 4$$
$$= 58\%$$

31 A 100% market share (or anything close) is highly unlikely to be achievable.

The increase in market share is highly unlikely to increase linearly with extra spending on marketing – it is much more likely there will be an ever slower rate of increase.

32 An exponential model tending to an asymptote below 100%, e.g. of the form $y = c - a^{-x}$, where $c < 100$.

33 a Substitute $\mathrm{g}(x)$ into $\mathrm{f}(x)$:

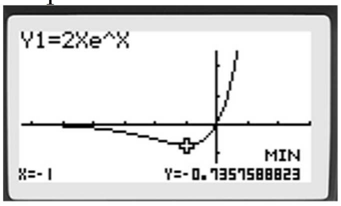
$$\mathrm{f}(\mathrm{g}(x)) = \frac{1}{(3x-4)-2}$$
$$= \frac{1}{3x-6}$$

b Substitute $\mathrm{f}(x)$ into $\mathrm{g}(x)$:

$$\mathrm{g}(\mathrm{f}(x)) = 3\left(\frac{1}{x-2}\right) - 4$$
$$= \frac{3}{x-2} - 4$$

34 Domain of f is $x \leq 2$ so domain of fg is
$x - 3 \leq 2$
$x \leq 5$

35 Graph the function and use 'min' to find the coordinates of the minimum point:



The turning point has x-coordinate $x = -1$, so largest possible domain of given form is $x \geq -1$.

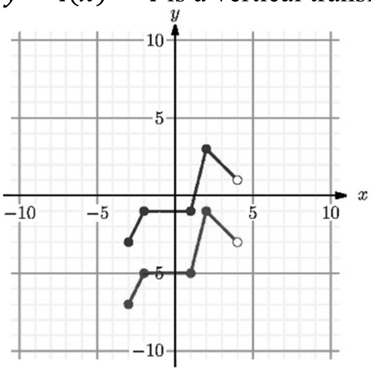
36 Let $y = \mathrm{f}(x)$ and rearrange to make x the subject:

$$y = \frac{x-1}{x+2}$$
$$xy + 2y = x - 1$$
$$x - xy = 1 + 2y$$
$$x(1-y) = 1 + 2y$$
$$x = \frac{1+2y}{1-y}$$

So,

$$\mathrm{f}^{-1}(x) = \frac{1+2x}{1-x}$$

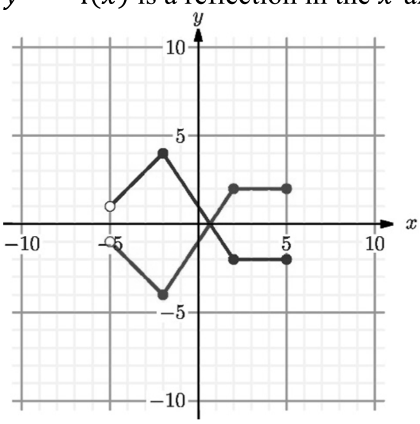
37 $y = \mathrm{f}(x) - 4$ is a vertical translation by -4:



38 A translation 3 units to the right is given by $y = \mathrm{f}(x-3)$:

$$y = (x-3)^2 - 2(x-3) + 5$$
$$= x^2 - 6x + 9 - 2x + 6 + 5$$
$$= x^2 - 8x + 20$$

39 $y = -\mathrm{f}(x)$ is a reflection in the x-axis:



40 A reflection in the y-axis is given by $y = \mathrm{f}(-x)$:

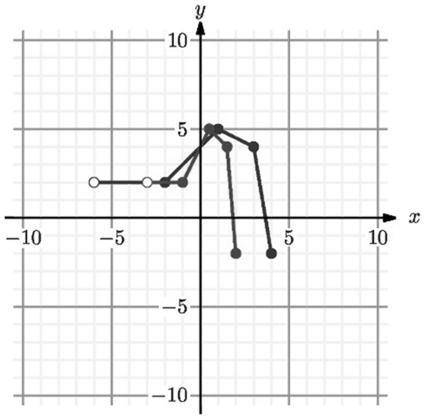
$$y = (-x)^3 + 3(-x)^2 - 4(-x) + 1$$
$$= -x^3 + 3x^2 + 4x + 1$$

41 A vertical stretch by scale factor 2 is given by $y = 2f(x)$:

$y = 2(3x^2 + x - 2)$

$= 6x^2 + 2x - 4$

42 $y = \mathrm{f}(2x)$ is a horizontal stretch with scale factor $\frac{1}{2}$:

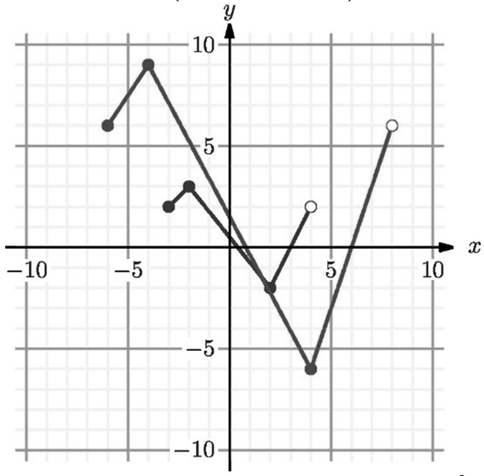


43 $y = 4f(x) + 1$ is a vertical stretch with scale factor 4 followed by a vertical translation by 1.

So, $(3, -2) \longrightarrow (3, -2 \times 4 + 1) = (3, -7)$

Note that the order matters for two vertical transformations.

44 $y = 3\mathrm{f}\left(\frac{1}{2}x\right)$ is vertical stretch with scale factor 3 and a horizontal stretch with scale factor 2 (in either order).



45 The model will be of the form $R = A\mathrm{e}^{kt} + d$, where d is the background radiation level and $k = -\frac{\ln 2}{t_{1/2}}$:

$R = A\mathrm{e}^{kt} + 3$

$k = -\frac{\ln 2}{15} \approx -0.0462$

When $t = 0, R = 45$:

$45 = A + 3$

$A = 42$

So, $R = 42\mathrm{e}^{-0.0462t} + 3$

46 A general sinusoidal model is of the form $T = a\sin\big(b(t-c)\big) + d$, where the amplitude is a, the period is $\frac{2\pi}{b}$, the phase shift is c and the central line is d:

$a = \frac{26-12}{2} = 7$

$b = \frac{2\pi}{24} \approx 0.262$

$d = \frac{26+12}{2} = 19$

The maximum for a sinusoidal model with period 24 hours but without phase shift would be at $t = 6$:

$c = 15.5 - 6 = 9.5$

So, $T = 7\sin\big(0.262(t-9.5)\big) + 19$

47 The model is of the form $N = \frac{L}{1+Ce^{-kt}}$ where L is the carrying capacity:

$N = \frac{600}{1+Ce^{-kt}}$

When $t = 0$, $N = 20$:

$$20 = \frac{600}{1+C}$$

$C = 29$

When $t = 2, N = 50$:

$50 = \frac{600}{1+29e^{-2k}}$

$29e^{-2k} = 11$

$$-2k = \ln\frac{11}{29}$$

$k \approx 0.485$

So, $N = \frac{600}{1+29e^{-0.485t}}$

48 The value of the function at the boundary ($x = 3$) must be the same on both branches:

$f(3) = 3^3 - k \times 3^2 + 1 = 28 - 9k$

and

$f(3) = k \times 3 + 4 = 3k + 4$

So, $28 - 9k = 3k + 4$

$12k = 24$

$k = 2$

49 Take logs to any base of both sides, and then use the laws of logs to rearrange into the form $y = mx + c$:

$$\begin{aligned}\log_{10} y &= \log_{10}(2x^{-3})\\ &= \log_{10} 2 + \log_{10} x^{-3}\\ &= \log_{10} 2 - 3\log_{10} x\end{aligned}$$

The graph of $\log_{10} y$ against $\log_{10} x$ is a straight line with gradient -3 and y-intercept $\log_{10} 2$.

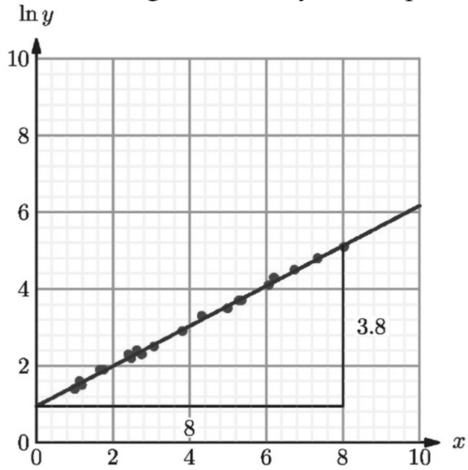
50 Take logs to any base of both sides, and then use the laws of logs to rearrange into the form $y = mx + c$:

$$\begin{aligned}\ln y &= \ln(5 \times 2^x)\\ &= \ln 5 + \ln 2^x\\ &= \ln 5 + x\ln 2\end{aligned}$$

The graph of $\ln y$ against x is a straight line with gradient $\ln 2$ and y-intercept $\ln 5$.

51 The semi-log graph looks to be closer to a straight line, so the relationship will be of the form $y = ka^x$.

Find the gradient and y-intercept of the semi-log graph:



gradient $\approx \frac{3.8}{8.0} = 0.475$

y-intercept ≈ 1

Since $\ln y = \ln k + x \ln a$, the gradient is $\ln a$ and the y-intercept is $\ln k$:

$\ln k \approx 1 \Rightarrow k \approx 2.7$

$\ln a \approx 0.475 \Rightarrow a \approx 1.6$

So, $y = 2.7 \times 1.6^x$

3 Geometry and trigonometry

1 $d = \sqrt{(7-2)^2 + (3-(-4))^2 + (-1-5)^2}$
$= \sqrt{25 + 49 + 36}$
$= 10.5$

2 $M = \left(\frac{1+(-5)}{2}, \frac{8+2}{2}, \frac{-3+4}{2}\right)$
$= (-2, 5, 0.5)$

3 Radius = 8 cm

Volume $= \frac{4}{3}\pi r^3 \quad = \frac{4}{3}\pi \times 8^3$

$= 2140 \text{ cm}^3$

Surface area $= 4\pi r^2 \quad = 4\pi \times 8^2$

$= 804 \text{ cm}^2$

4 Volume $= \frac{1}{3}\pi r^2 h$

$= \frac{1}{3}\pi \times 6^2 \times 15$

$= 565 \text{ cm}^3$

Slope length, l, is given by:

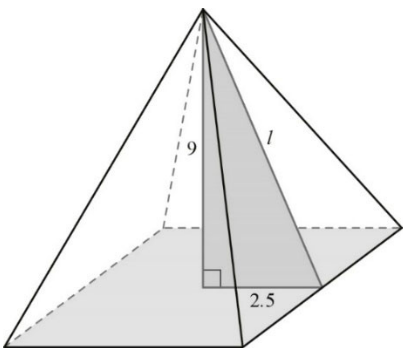
$l = \sqrt{r^2 + h^2}$
$= \sqrt{6^2 + 15^2}$
$= 3\sqrt{29}$

Surface area $= \pi r l + \pi r^2$

$= \pi \times 6 \times 3\sqrt{29} + \pi \times 6^2$

$= 418 \text{ cm}^2$

5 Volume $= \frac{1}{3}x^2 h \qquad = \frac{1}{3} \times 5^2 \times 9 \ = 75.0 \text{ cm}^3$



$l = \sqrt{2.5^2 + 9^2}$
$= \frac{\sqrt{349}}{2}$

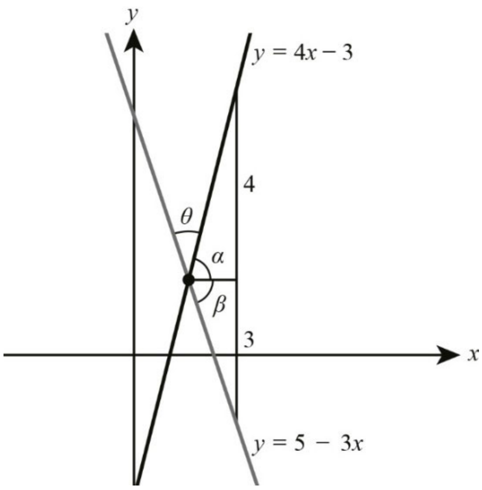
Surface area $= 5^2 + 4\left(\frac{1}{2} \times 5 \times \frac{\sqrt{349}}{2}\right) \ = 118 \text{ cm}^2$

6 Volume $= \pi r^2 h + \frac{2}{3}\pi r^3$

$= \pi \times 5^2 \times 30 + \frac{2}{3}\pi \times 5^3$

$= 2620\ \mathrm{m}^3$

7 Draw the lines and label the angle required θ:



Since the gradient of $y = 4x - 3$ is 4,

$\tan\alpha = \frac{4}{1}$

$\alpha = \tan^{-1} 4 = 76.0°$

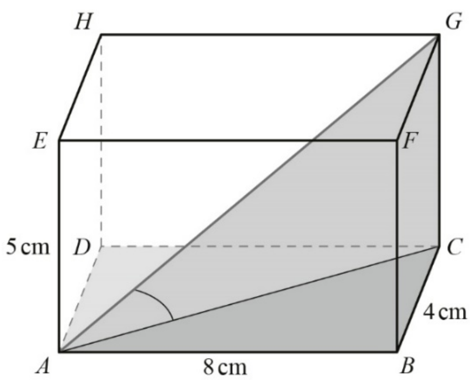
Since the gradient of $y = 5 - 3x$ is -3,

$\tan\beta = \frac{3}{1}$

$\alpha = \tan^{-1} 3 = 71.6°$

So, $\theta = 180 - 76.0 - 71.6 = 32.4°$

8 Draw in the diagonal AG. Angle needed is $G\hat{A}C = \theta$



First work in triangle ABC:

$AC = \sqrt{8^2 + 4^2}$

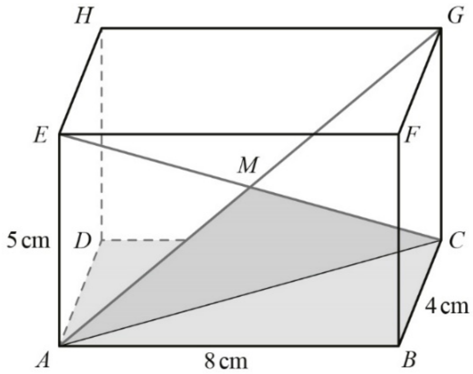
$= \sqrt{80}$

Then in triangle ACG:

$\tan\theta = \frac{5}{\sqrt{80}}$

$\theta = \tan^{-1}\frac{5}{\sqrt{80}} = 29.2°$

9 Draw in the two diagonals – they will intersect at the midpoint of each, M.



From triangle AGC:

$$AG = \sqrt{\left(\sqrt{80}\right)^2 + 5^2}$$
$$= \sqrt{105}$$

By symmetry, $EC = \sqrt{105}$

And $AM = CM = \frac{\sqrt{105}}{2}$

Using the cosine rule in triangle AMC:

$$\cos M = \frac{AM^2 + CM^2 - AC^2}{2(AM)(CM)}$$
$$= \frac{26.25 + 26.25 - 80}{52.5}$$
$$M = 121.6°$$

So, acute angle between AG and EC is $180 - 121.6 = 58.4°$

10 $\sin\theta = \frac{1.8}{4.9}$

$$\theta = \sin^{-1}\frac{1.8}{4.9}$$
$$= 21.6°$$

11 By the sine rule,

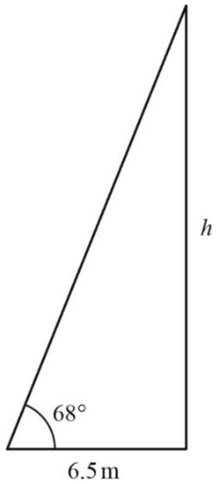
$$\frac{3.8}{\sin 80} = \frac{AC}{\sin 55}$$
$$AC = \frac{3.8}{\sin 80} \times \sin 55$$
$$= 3.16 \text{ cm}$$

12 By the cosine rule,

$$\cos C = \frac{a^2 + b^2 - c^2}{2ab}$$
$$= \frac{12^2 + 10^2 - 9^2}{2 \times 12 \times 10}$$
$$C = \cos^{-1}\left(\frac{12^2 + 10^2 - 9^2}{2 \times 12 \times 10}\right) = 47.2°$$

13 $A = \frac{1}{2} \times 6 \times 15 \times \sin 42$
$= 30.1$

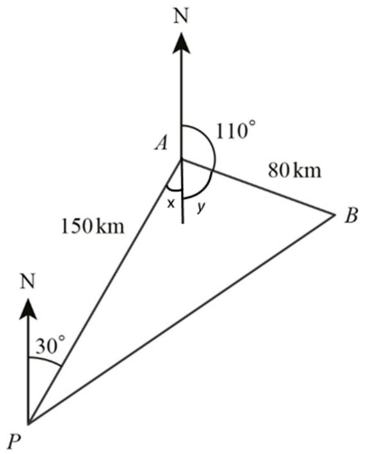
14 Draw a diagram:



$\tan 68 = \frac{h}{6.5}$

$h = 6.5 \tan 68$
$= 16.1 \text{ m}$

15 Start by drawing the situation described:



$x = 30°$ by alternate angles

$y = 180 - 110 = 70°$

So, $P\hat{A}B = 30 + 70 = 100°$

By the cosine rule,

$d^2 = 150^2 + 80^2 - 2 \times 150 \times 80 \cos 100$

$d = \sqrt{150^2 + 80^2 - 2 \times 150 \times 80 \cos 100}$
$= 182 \text{ km}$

16 $s = \frac{\theta}{360} \times 2\pi r$
$= \frac{55}{360} \times 2\pi \times 6$
$= 5.76 \text{ cm}$

17 $A = \frac{\theta}{360} \times \pi r^2$

$= \frac{110}{360} \times \pi \times 10^2$

$= 96.0 \text{ cm}^2$

18 Rearrange the equation to find the gradient of the line segment AB:

$$2x + 3y = 5$$
$$3y = -2x + 5$$
$$y = -\frac{2}{3}x + \frac{5}{3}$$

The gradient of AB is $-\frac{2}{3}$

So the gradient of the perpendicular to AB is $-\frac{1}{\left(-\frac{2}{3}\right)} = \frac{3}{2}$

The perpendicular bisector has gradient $\frac{3}{2}$ and passes through the point $(4, 7)$:

$$y - 7 = \frac{3}{2}(x - 4)$$

$$y = \frac{3}{2}x + 1$$

19 The midpoint of $(-3, -2)$ and $(1, 8)$ is $\left(\frac{-3+1}{2}, \frac{-2+8}{2}\right) = (-1, 3)$

The gradient of the line segment between $(-3, -2)$ and $(1, 8)$ is $\frac{8-(-2)}{1-(-3)} = \frac{5}{2}$

So the gradient of the perpendicular bisector is $-\frac{1}{\left(\frac{5}{2}\right)} = -\frac{2}{5}$

The perpendicular bisector has gradient $-\frac{2}{5}$ and passes through the point $(-1, 3)$:

$$y - 3 = -\frac{2}{5}(x + 1)$$

$$y = -\frac{2}{5}x + \frac{13}{5}$$

20 a The sites are the labelled points:

A, B, C, D, E

b The vertices are the points where the edges meet:

P, Q, R, S, T

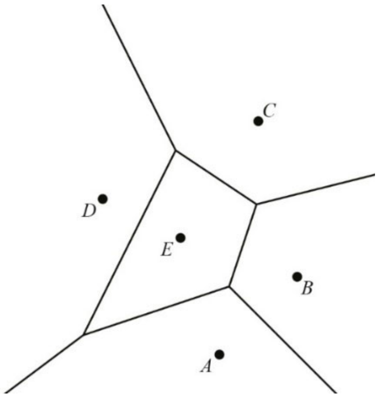
c The finite edges are the lines that have vertices at both ends:

PQ, QR, RS, ST, PT, QT

d The finite cells are the fully enclosed areas:

$PTQ, TQRS$

21

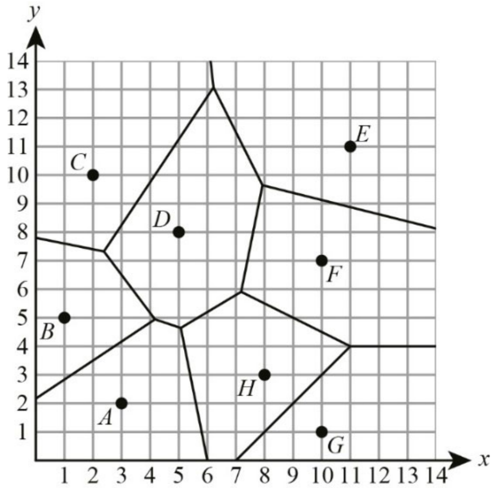


Apply the incremental algorithm starting with the perpendicular bisector of DE, since E is in cell D.

Then travel along the perpendicular bisector of CE, them BE, then AE.

That completes the cell E so delete all unused perpendicular bisectors:

22



$(5, 4)$ lies in cell A, so by nearest neighbour interpolation, estimate the value of the function to be 12.

23 The possible locations are the four vertices. Of these, the one furthest from any town is clearly CDE.

Find the coordinates of vertex CDE as the intersection of the perpendicular bisectors of CD and DE.

Midpoint of C and D is $(6, 7)$ and the perpendicular between them has gradient -0.5.

Bisector CD has equation:

$y - 7 = -0.5(x - 6)$
$2y + x = 20$ **(1)**

Midpoint of D and E is $(6.5, 3)$ and the perpendicular between them has gradient 0.75.

Bisector of DE has equation

$y - 3 = 0.75(x - 6.5)$
$6x - 8y = 15$ **(2)**

The intersection of these lines is vertex CDE:

$(2) + 4(1)$: $10x = 95$

CDE has coordinates $(9.5, 5.25)$

24 a $55° = 55 \times \frac{2\pi}{360} = \frac{11\pi}{36}$ radians

b 1.2 radians $= 1.2 \times \frac{360}{2\pi} = 68.8°$

25 $s = r\theta$
$= 6 \times 0.7$

$= 4.2$ cm

26 $A = \frac{1}{2}r^2\theta$
$= \frac{1}{2} \times 10^2 \times 1.8$
$= 90 \text{ cm}^2$

27 Using the unit circle:

a $\sin(\theta + \pi) = -\sin\theta = -0.4$

b $\cos\left(\theta - \frac{\pi}{2}\right) = \sin\theta = 0.4$

28 Use the identity $\cos^2\theta + \sin^2\theta \equiv 1$ to relate the value of $\cos\theta$ to the value of $\sin\theta$:

$\cos^2\theta = 1 - \sin^2\theta$
$= 1 - \frac{9}{16} \quad = \frac{7}{16}$

$\cos\theta = \pm\frac{\sqrt{7}}{4}$

But $\cos\theta < 0$ for $\frac{\pi}{2} < \theta < \pi$

So, $\cos\theta = -\frac{\sqrt{7}}{4}$

29 $\tan(2\pi - \theta) = \frac{\sin(2\pi-\theta)}{\cos(2\pi-\theta)}$
$= \frac{\sin(-\theta)}{\cos(-\theta)}$ (since sin and cos are 2π periodic)
$= \frac{-\sin\theta}{\cos\theta}$
$= -\tan\theta$

Note that certain relationships, such as $\sin(-\theta) = -\sin\theta$ and $\cos(-\theta) = \cos\theta$, are used so often that you should just know them. You do not want to have to go back to the unit circle each time to derive them.

30 By the sine rule,

$\frac{\sin\theta}{14} = \frac{\sin 38}{11}$
$\theta = \sin^{-1}\left(\frac{\sin 38}{11} \times 14\right)$

$\theta = 51.6°$ or $\theta = 180 - 51.6 = 128.4°$

Check that each value of θ is possible by making sure that the angle sum in each case is less than 180°:

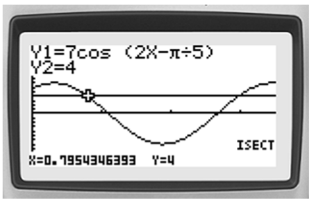
$51.6 + 38 = 89.6 < 180$

$128.4 + 38 = 166.4 < 180$

So, both are possible:

$\theta = 51.6°$ or $128.4°$

31 Graph $y = 7\cos\left(2x - \frac{\pi}{5}\right)$ and $y = 4$ and find the x-values of the intersection points:



$x = 0.795, 2.97$

32 The mirror line is of the form $y = (\tan\theta)x$:

$\tan\theta = \frac{1}{2} \Rightarrow \theta = 26.565°\ldots$

Multiply the reflection matrix by the original point:

$$\begin{pmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{pmatrix}\begin{pmatrix} 3 \\ -2 \end{pmatrix} = \begin{pmatrix} 0.6 & 0.8 \\ 0.8 & -0.6 \end{pmatrix}\begin{pmatrix} 3 \\ -2 \end{pmatrix} = \begin{pmatrix} 0.2 \\ 3.6 \end{pmatrix}$$

So, the image is $(0.2, 3.6)$

33 The matrix for a horizontal stretch with scale factor k is $\begin{pmatrix} k & 0 \\ 0 & 1 \end{pmatrix}$:

$$\begin{pmatrix} \frac{1}{3} & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 12 \\ 5 \end{pmatrix} = \begin{pmatrix} 4 \\ 5 \end{pmatrix}$$

So, the image is $(4, 5)$.

34 The matrix for a vertical stretch with scale factor k is $\begin{pmatrix} 1 & 0 \\ 0 & k \end{pmatrix}$:

$$\begin{pmatrix} 1 & 0 \\ 0 & 3.8 \end{pmatrix}\begin{pmatrix} -1 \\ 4 \end{pmatrix} = \begin{pmatrix} -1 \\ 15.2 \end{pmatrix}$$

So, the image is $(-1, 15.2)$

35 The matrix for an enlargement with scale factor k is $\begin{pmatrix} k & 0 \\ 0 & k \end{pmatrix}$:

$$\begin{pmatrix} 2.5 & 0 \\ 0 & 2.5 \end{pmatrix}\begin{pmatrix} 3 \\ -10 \end{pmatrix} = \begin{pmatrix} 7.5 \\ -25 \end{pmatrix}$$

So, the image is $(7.5, -25)$

36 The matrix for an anticlockwise rotation is $\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$:

$$\begin{pmatrix} \cos 60 & -\sin 60 \\ \sin 60 & \cos 60 \end{pmatrix}\begin{pmatrix} 4 \\ -6 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}\begin{pmatrix} 4 \\ -6 \end{pmatrix} = \begin{pmatrix} 2 + 3\sqrt{3} \\ 2\sqrt{3} - 3 \end{pmatrix}$$

So, the image is $\left(2 + 3\sqrt{3}, 2\sqrt{3} - 3\right)$

37 The matrix for a clockwise rotation is $\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$:

$$\begin{pmatrix} \cos 45 & \sin 45 \\ -\sin 45 & \cos 45 \end{pmatrix}\begin{pmatrix} 2 \\ 8 \end{pmatrix} = \begin{pmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{pmatrix}\begin{pmatrix} 2 \\ 8 \end{pmatrix} = \begin{pmatrix} 5\sqrt{2} \\ 3\sqrt{2} \end{pmatrix}$$

So, the image is $\left(5\sqrt{2}, 3\sqrt{2}\right)$

38 $\begin{pmatrix}-9\\5\end{pmatrix}+\begin{pmatrix}6\\-2\end{pmatrix}=\begin{pmatrix}-3\\3\end{pmatrix}$

So, the image is $(-3, 3)$

39 First apply the reflection matrix:

$\tan\theta = -1$
$\theta = -45°$

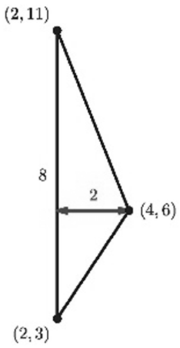
$\begin{pmatrix}\cos 2\theta & \sin 2\theta\\ \sin 2\theta & -\cos 2\theta\end{pmatrix}\begin{pmatrix}3\\7\end{pmatrix}=\begin{pmatrix}0 & -1\\-1 & 0\end{pmatrix}\begin{pmatrix}3\\7\end{pmatrix}=\begin{pmatrix}-7\\-3\end{pmatrix}$

Then apply the rotation matrix:

$\begin{pmatrix}\cos 90 & -\sin 90\\ \sin 90 & \cos 90\end{pmatrix}\begin{pmatrix}-7\\-3\end{pmatrix}=\begin{pmatrix}0 & -1\\1 & 0\end{pmatrix}\begin{pmatrix}-7\\-3\end{pmatrix}=\begin{pmatrix}3\\-7\end{pmatrix}$

So, the image is $(3, -7)$

40 Find the area of the original triangle:



Area $=\frac{1}{2}\times 8\times 2$
$= 8$

Then use the result that Area of image $= |\det M| \times$ area of original:

Area of image $= |-6-4|\times 8$
$= 80$

41 The vector goes 5 units to the left and 2 units up:

$\mathbf{v}=\begin{pmatrix}-5\\2\end{pmatrix}$

42 $\begin{pmatrix}4\\-2\\1\end{pmatrix}=4\mathbf{i}-2\mathbf{j}+\mathbf{k}$

43 $\mathbf{i}-6\mathbf{k}=\begin{pmatrix}1\\0\\-6\end{pmatrix}$

44 Trace a path from P to S via Q and R, noting that going backwards along an arrow means the vector needs to be negative:

$\overrightarrow{PS}=\overrightarrow{PQ}+\overrightarrow{QR}+\overrightarrow{RS}$
$=\mathbf{a}+\mathbf{b}-\mathbf{c}$

45 Since M is the midpoint of YZ, $\overrightarrow{YM} = \frac{1}{2}\overrightarrow{YZ}$ so start by finding an expression for $\overrightarrow{YZ}$:

$$\overrightarrow{YZ} = \overrightarrow{YX} + \overrightarrow{XZ}$$
$$= -\mathbf{a} + \mathbf{b}$$
$$\overrightarrow{XM} = \overrightarrow{XY} + \frac{1}{2}\overrightarrow{YZ}$$
$$= \mathbf{a} + \frac{1}{2}(-\mathbf{a} + \mathbf{b})$$
$$= \frac{1}{2}\mathbf{a} + \frac{1}{2}\mathbf{b}$$

46 If two vectors are parallel then $\mathbf{b} = t\mathbf{a}$ for some scalar t:

$$\begin{pmatrix} p \\ q \\ -12 \end{pmatrix} = t\begin{pmatrix} -1 \\ 2 \\ 4 \end{pmatrix}$$
$$= \begin{pmatrix} -t \\ 2t \\ 4t \end{pmatrix}$$
$$\begin{cases} p = -t & (1) \\ q = 2t & (2) \\ -12 = 4t & (3) \end{cases}$$

From (3): $t = -3$

So, $p = 3, q = -6$

47 $\overrightarrow{AB} = \mathbf{b} - \mathbf{a}$

$$= \begin{pmatrix} -4 \\ 3 \\ -1 \end{pmatrix} - \begin{pmatrix} 5 \\ 1 \\ -2 \end{pmatrix}$$
$$= \begin{pmatrix} -9 \\ 2 \\ 1 \end{pmatrix}$$

48 $|\mathbf{v}| = \sqrt{3^2 + (-5)^2 + (-1)^2}$

$$= \sqrt{35}$$

49 Unit vector $= \frac{1}{\sqrt{2^2+6^2+(-3)^2}}\begin{pmatrix} 2 \\ 6 \\ -3 \end{pmatrix}$

$$= \frac{1}{7}\begin{pmatrix} 2 \\ 6 \\ -3 \end{pmatrix}$$

50 Use $\mathbf{r} = \mathbf{a} + \lambda\mathbf{d}$ with $\mathbf{a} = \begin{pmatrix} 5 \\ 4 \\ -7 \end{pmatrix}$ and $\mathbf{d} = \begin{pmatrix} -1 \\ 2 \\ 0 \end{pmatrix}$:

$$\mathbf{r} = \begin{pmatrix} 5 \\ 4 \\ -7 \end{pmatrix} + \lambda\begin{pmatrix} -1 \\ 2 \\ 0 \end{pmatrix}$$

51 A direction vector for the line is given by the vector $\overrightarrow{AB} = \mathbf{b} - \mathbf{a}$:

$$\mathbf{d} = \begin{pmatrix} 3 \\ -5 \\ 4 \end{pmatrix} - \begin{pmatrix} -2 \\ 6 \\ 1 \end{pmatrix} = \begin{pmatrix} 5 \\ -11 \\ 3 \end{pmatrix}$$

Use $\mathbf{r} = \mathbf{a} + \lambda\mathbf{d}$:

$$\mathbf{r} = \begin{pmatrix} -2 \\ 6 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 5 \\ -11 \\ 3 \end{pmatrix}$$

52 Write $\mathbf{r}$ as $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ and form a separate equation for each component:

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -3 \\ 0 \\ 5 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 4 \\ -2 \end{pmatrix} = \begin{pmatrix} -3 + \lambda \\ 4\lambda \\ 5 - 2\lambda \end{pmatrix}$$

So, $x = -3 + \lambda,\ y = 4\lambda,\ z = 5 - 2\lambda$

53 Set the position vectors to be equal and solve for λ and μ:

$$\begin{pmatrix} -1 \\ -7 \\ 5 \end{pmatrix} + \lambda \begin{pmatrix} -3 \\ 1 \\ 4 \end{pmatrix} = \begin{pmatrix} 2 \\ -3 \\ -6 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$$

$$\begin{cases} -1 - 3\lambda = 2 + \mu & (1) \\ -7 + \lambda = -3 - 2\mu & (2) \\ 5 + 4\lambda = -6 + \mu & (3) \end{cases}$$

Solve any pair simultaneously, say (1) and (2):

$$\begin{cases} 3\lambda + \mu = -3 \\ \lambda + 2\mu = 4 \end{cases}$$

From GDC: $\lambda = -2, \mu = 3$

Check in (3):

$5 + 4(-2) = -3$

$-6 + 3 = -3$

So, the lines intersect.

Note that the values of λ and μ have to work in all three equations for the lines to intersect.

Substitute $\lambda = -2$ into (1) (or $\mu = 3$ into (2)) to find the point of intersection:

$$\mathbf{r} = \begin{pmatrix} -1 \\ -7 \\ 5 \end{pmatrix} - 2 \begin{pmatrix} -3 \\ 1 \\ 4 \end{pmatrix} = \begin{pmatrix} 5 \\ -9 \\ -3 \end{pmatrix}$$

So, point of intersection is $(5, -9, -3)$

54 The required velocity vector will have magnitude 32.5 and be in the direction of $\begin{pmatrix} 3 \\ -4 \\ 12 \end{pmatrix}$:

$$\text{unit vector} = \frac{1}{\sqrt{3^2 + (-4)^2 + 12^2}} \begin{pmatrix} 3 \\ -4 \\ 12 \end{pmatrix}$$
$$= \frac{1}{13} \begin{pmatrix} 3 \\ -4 \\ 12 \end{pmatrix}$$

So, $\quad \mathbf{v} = 32.5 \times \frac{1}{13} \begin{pmatrix} 3 \\ -4 \\ 12 \end{pmatrix}$
$$= \begin{pmatrix} 7.5 \\ -10 \\ 30 \end{pmatrix} \text{ ms}^{-1}$$

55 The position vector at time t is given by $\mathbf{r} = \mathbf{r}_0 + t\mathbf{v}$:

$$\mathbf{r} = -5\mathbf{i} + \mathbf{k} + 10(3\mathbf{i} - \mathbf{j} + 2\mathbf{k})$$
$$= 25\mathbf{i} - 10\mathbf{j} + 21\mathbf{k}$$

56 If the drones did collide, their position vectors would be equal for some time, t:

$$\begin{pmatrix} 3 \\ -5 \\ 1 \end{pmatrix} + t \begin{pmatrix} 2 \\ 4 \\ 3 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix} + t \begin{pmatrix} 5 \\ -8 \\ 1 \end{pmatrix}$$

$$\begin{cases} 3 + 2t = 1 + 5t & (1) \\ -5 + 4t = 3 - 8t & (2) \\ 1 + 3t = 2 + t & (3) \end{cases}$$

From (1): $3t = 2 \Rightarrow t = \frac{2}{3}$

From (2): $12t = 8 \Rightarrow t = \frac{2}{3}$

From (3): $2t = 1 \Rightarrow t = \frac{1}{2} \neq \frac{2}{3}$

Since there is no common time when the position vectors are equal, the drones do not collide.

57 a Differentiate to find the acceleration vector:

$$\mathbf{a} = \begin{pmatrix} 3\mathrm{e}^t \\ -8\mathrm{e}^{-2t} \end{pmatrix}$$

b Integrate to find the displacement:

$$\mathbf{r} = \begin{pmatrix} 3\mathrm{e}^t + c_1 \\ -2\mathrm{e}^{-2t} + c_2 \end{pmatrix}$$

When $t = 0, \mathbf{r} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$:

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 3 + c_1 \\ -2 + c_2 \end{pmatrix}$$

$$\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} -3 \\ 2 \end{pmatrix}$$

So, $\mathbf{r} = \begin{pmatrix} 3\mathrm{e}^t - 3 \\ -2\mathrm{e}^{-2t} + 2 \end{pmatrix}$

58 The projectile will reach its maximum height when the vertical component of its velocity is zero:

$\mathbf{v} = 12\mathbf{i} + (15 - 9.8t)\mathbf{j}\ \text{ms}^{-1}$

$15 - 9.8t = 0$

$t = \dfrac{15}{9.8}$

The maximum height is the vertical component of the position vector at this time:

$h_{max} = 15\left(\dfrac{15}{9.8}\right) - 4.9\left(\dfrac{15}{9.8}\right)^2$
$= 11.5\ \text{m}$

59 a The distance between two points is $|\mathbf{b} - \mathbf{a}|$:

$d = \left|\begin{pmatrix} 2\cos 3t + 1 \\ 2\sin 3t - 5 \end{pmatrix} - \begin{pmatrix} 1 \\ -5 \end{pmatrix}\right|$
$= \left|\begin{matrix} 2\cos 3t \\ 2\sin 3t \end{matrix}\right|$
$= \sqrt{4\cos^2 3t + 4\sin^2 3t}$
$= \sqrt{4(\cos^2 3t + \sin^2 3t)}$
$= \sqrt{4}$ (since $\sin^2\theta + \cos^2\theta \equiv 1$)
$= 2$

b So, the object moves in a circle of radius 2, centre $(1, -5)$

60 $\mathbf{a}\cdot\mathbf{b} = a_1b_1 + a_2b_2 + a_3b_3$
$= (-4)(2) + (3)(5) + (1)(-3)$
$= 4$

Note that although this could be done on the GDC, it is important to be able to do the calculation by hand as well.

61 $\mathbf{a}\cdot\mathbf{b} = |\mathbf{a}||\mathbf{b}|\cos\theta$
$= (3)(8)\cos 45°$
$= 24\left(\dfrac{\sqrt{2}}{2}\right)$
$= 12\sqrt{2}$

62 $\cos\theta = \frac{a_1b_1 + a_2b_2 + a_3b_3}{|\mathbf{a}||\mathbf{b}|}$
$= \dfrac{(1)(4) + (-3)(-2) + (2)(-7)}{\sqrt{1^2 + (-3)^2 + 2^2}\sqrt{4^2 + (-2)^2 + (-7)^2}}$
$= \dfrac{-4}{\sqrt{14}\sqrt{69}}$

$\theta = \cos^{-1}\left(\frac{-4}{\sqrt{14}\sqrt{69}}\right) = 97.4°$

Required angle is acute so, $180 - 97.4 = 82.6°$

63 For two vectors to be perpendicular, their scalar product must be zero:

$\begin{pmatrix} 2 + t \\ -3 \\ t \end{pmatrix}\cdot\begin{pmatrix} 1 \\ 4t - 1 \\ 5 \end{pmatrix} = 0$
$2 + t - 3(4t - 1) + 5t = 0$
$5 - 6t = 0$
$t = \dfrac{5}{6}$

64 The angle between the lines is the angle between their direction vectors:

$$\cos\theta = \frac{(2)(1)+(1)(3)+(-1)(-5)}{\sqrt{2^2+1^2+(-1)^2}\sqrt{1^2+3^2+(-5)^2}}$$

$$= \frac{10}{\sqrt{6}\sqrt{35}}$$

$$\theta = \cos^{-1}\left(\frac{10}{\sqrt{6}\sqrt{35}}\right)$$

$$= 46.4°$$

65 Use the result $\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \times \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = \begin{pmatrix} a_2b_3 - b_2a_3 \\ a_3b_1 - b_3a_1 \\ a_1b_2 - b_1a_2 \end{pmatrix}$:

$$\begin{pmatrix} 1 \\ 4 \\ 3 \end{pmatrix} \times \begin{pmatrix} 5 \\ 2 \\ -2 \end{pmatrix} = \begin{pmatrix} (4)(-2)-(2)(3) \\ (3)(5)-(-2)(1) \\ (1)(2)-(5)(4) \end{pmatrix}$$

$$= \begin{pmatrix} -14 \\ 17 \\ -18 \end{pmatrix}$$

Note that although this could be done on the GDC, it is important to be able to do the calculation by hand as well.

66 $|\mathbf{a} \times \mathbf{b}| = |\mathbf{a}||\mathbf{b}|\sin\theta$

$$= (4)(5)\sin 30°$$

$$= 20\left(\frac{1}{2}\right)$$

$$= 10$$

67 Any two vectors (starting at the same vertex) are suitable for defining the triangle, for example $\overrightarrow{PQ}$ and $\overrightarrow{PR}$:

$$\overrightarrow{PQ} = \begin{pmatrix} 4 \\ -2 \\ 5 \end{pmatrix} - \begin{pmatrix} -1 \\ 3 \\ -2 \end{pmatrix} = \begin{pmatrix} 5 \\ -5 \\ 7 \end{pmatrix}$$

$$\overrightarrow{PR} = \begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix} - \begin{pmatrix} -1 \\ 3 \\ -2 \end{pmatrix} = \begin{pmatrix} 2 \\ -3 \\ 5 \end{pmatrix}$$

$$\overrightarrow{PQ} \times \overrightarrow{PR} = \begin{pmatrix} 5 \\ -5 \\ 7 \end{pmatrix} \times \begin{pmatrix} 2 \\ -3 \\ 5 \end{pmatrix}$$

$$= \begin{pmatrix} -4 \\ -11 \\ -5 \end{pmatrix}$$

$$\text{Area} = \frac{1}{2}\left|\overrightarrow{PQ} \times \overrightarrow{PR}\right|$$

$$= \frac{1}{2}\sqrt{(-4)^2+(-11)^2+(-5)^2}$$

$$= \frac{9\sqrt{2}}{2}$$

68 a The component of vector **a** acting in the direction of vector **b** is $\frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{b}|}$:

$$\frac{(10)(2)+(-3)(1)+(4)(-2)}{\sqrt{2^2+1^2+(-2)^2}} = 3$$

b The component of vector **a** acting perpendicular to vector **b** is $\frac{|\mathbf{a}\times\mathbf{b}|}{|\mathbf{b}|}$:

$$\frac{|(10\mathbf{i}-3\mathbf{j}+4\mathbf{k})\times(2\mathbf{i}+\mathbf{j}-2\mathbf{k})|}{|2\mathbf{i}+\mathbf{j}-2\mathbf{k}|} = \frac{|2\mathbf{i}+28\mathbf{j}+16\mathbf{k}|}{\sqrt{2^2+1^2+(-2)^2}}$$

$$= \frac{\sqrt{2^2+28^2+16^2}}{3}$$

$$= 10.8$$

69 a A, C, D

b 2

70 a B, D (a simple graph has no loops or repeated edges)

b B, C (each vertex is connected to every other vertex)

71 a Yes

b It is not possible to get from e.g. A to B.

72 a 1

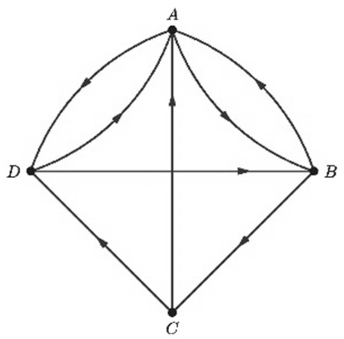
b 2

73 A and C (a tree has no closed paths)

74 Remember that the loop at B counts as two edges.

	A	B	C	D	E
A	0	0	0	1	2
B	0	2	1	2	0
C	0	1	0	1	0
D	1	2	1	0	0
E	2	0	0	0	0

75



76 a Raise the matrix to the power of 10 and look at the entry in row A and column C. There are 146 walks.

b Raise the matrix to the power of 5 and look at the entry in row and column D. There are 7 walks.

77

	A	B	C	D	E
A	-	46	-	-	39
B	46	-	28	57	68
C	-	28	-	-	47
D	-	57	-	-	52
E	39	68	47	52	-

78 a

0	1/3	1/3	0	½
½	0	1/3	0	0
0	1/3	0	½	0
0	0	1/3	0	½
1/2	1/3	0	½	0

b Look at the entry in the second row and second column in the matrix raised to the power of 8. The probability is 0.278.

79 Raise the transition matrix from Question 78 to a large power. Each column is

$$\begin{pmatrix} 0.238 \\ 0.169 \\ 0.147 \\ 0.182 \\ 0.266 \end{pmatrix}$$

The largest element is the last one, so the highest ranked page will be E.

80

	trail	**path**	**circuit**	**cycle**
BCDBE	Yes	No	No	No
BAFEB	No	Yes	Yes	Yes
CDBAFEC	No	Yes	Yes	Yes
CDBCEBAFE	Yes	No	No	No
BDCEBC	Yes	No	No	No

A trail has no repeated edges.

A path has no repeated vertices.

A circuit has no repeated edges and is closed.

A cycle has no repeated vertices (except for the start/end one) and is closed.

81 a No, because it has vertices of odd degree (C and E).

b C and E have odd degrees, so these need to be the start and end vertices; e.g. CDBCEFABE.

82 A Hamiltonian cycle needs to visit every vertex exactly once.

ABDCEFA, AFECDBA (This is the same cycle but in reverse.)

83 AB(3), DE(4), AC(5), AF(6), CD(8)

Weight = 26

84 FA(6), AB(3), AC(5), CD(8), DE(4)

Weight = 26

85 AE(4), EF(4), FC(3), AB(5), BD(5), DG(2)

Weight of tree = 23

86 a A and B have odd degrees, so the edge AB needs to be repeated. The length of the required route is 115.

b The route needs to start and finish at a vertex of odd degree, so it could start at A or B.

87 The vertices of odd degree are A, C, F, G. Consider all possible pairings:

AC + FG = 10 + 6 = 16

AF + CG = 10 + 11 = 21

AG + CF = 4 + 13 = 17

The edges AC and FG should be repeated.

The length of the required route is 77.

88 Notice that the shortest path from A to E is via B: AB + BE = 4 + 6 = 10.

You also need to find the shortest paths from A to C (ABC = 11), from A to D (ABD = 9), and from C to D (CED = 11).

	A	B	C	D	E
A	-	4	11	9	10
B	4	-	7	5	6
C	11	7	-	11	8
D	9	5	11	-	3
E	10	6	8	3	-

89 The shortest edges from each vertex, starting from A, not repeating any vertices (do not forget to complete the cycle):

AB(4), BD(5), DE(3), EC(8), CA(11)

So, an upper bound is 4 + 5 + 3 + 8 + 11 = 31

90 The minimum spanning tree for A, B, C, E can be found using Kruskal's algorithm:

AB(4), BE(6), BC(7)

The two shortest edges from D are DE(3) and DB(5).

So, a lower bound is 4 + 6 + 7 + 3 + 5 = 25

91 We want the upper bound to be as small as possible and the lower bound to be as large as possible.
So, $29 \leq L \leq 31$

4 Statistics and probability

1 a Discrete – it can only take certain values.

b Continuous (although its measurement might be discrete because it will be to a particular accuracy).

c Discrete.

2 a There are many possibilities. It could be all his patients, all his ill patients, all people in his area, or all people in the world.

b This is not a random sample of the population because there are some people (those who do not attend the clinic) who cannot possibly be included.

3 Yes – people who do less exercise might be less likely to choose to participate.

4 Yes – there seems to be consistency when the observation is repeated (within a reasonable statistical noise).

5 Item D is not a possible human height. It might have been a participant not taking the test seriously or it might have been a misread (e.g. giving height in feet and inches). You could either return to participant D and ask them to check their response, or if this were not possible you would discard the data item.

6 The IQR is 4. Anything above $11 + 1.5 \times 4 = 17$ is an outlier. Just because a data item is an outlier does not mean that it should be excluded. It should be investigated carefully to ensure that it is still a valid member of the population of interest.

7 Convenience sampling.

8 The proportion from Italy is $\frac{60}{150} = \frac{2}{5}$. The stratified sample must be in the same proportion, so it should contain $\frac{2}{5} \times 20 = 8$ students from Italy.

9 The proportion is $\frac{18+12}{15+18+12} = \frac{30}{45} = \frac{2}{3}$

10 This is all the 30–40 group and half of the 20–30 group, which is $7+\frac{4}{2} = 9$

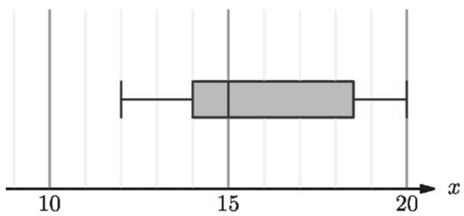
11 a The total frequency is 160. Half of this frequency (80) on the frequency axis reads as about 42 on the x-axis, which is the median.

b The lower quartile corresponds to a frequency of 40, which is an x-value of approximately 30.

The upper quartile corresponds to a frequency of 120 which is an x-value of approximately 60. Therefore, the interquartile range is $60 - 30 = 30$.

c The 90th percentile corresponds to a frequency of $0.9 \times 160 = 144$. This has an x-value of about 72, which is the 90th percentile.

12 Putting the data into the GDC, the following summary statistics can be found:



Min 12; lower quartile 14; median 16; upper quartile 18.5; max 20

13 a Both have a similar spread (same IQR (4) and range excluding outliers (10)) but population A is higher on average (median of 16 versus 10).

b B is more likely to be normally distributed as it has a symmetric distribution.

14 The mode is 14 as it is the only one which occurs twice. From the GD, the median is 18 and the mean is 18.5

Tip: You should also be able calculate the median without a calculator.

15 The mean is $\frac{23+x}{5} = 7$

So $23 + x = 35$, therefore $x = 12$

16 a The midpoints are 15, 25, 40, 55.

$n = 10 + 12 + 15 + 13 = 50$

$$\bar{x} = \frac{10 \times 15 + 12 \times 25 + 15 \times 40 + 13 \times 55}{50} = \frac{1765}{50} = 35.3$$

b We are not using the original data.

17 The modal class is $15 < x \leq 20$

18 a From the GDC, $Q_3 = 18, Q_1 = 7$ so IQR $= 11$

b From standard deviation is 5.45

c The variance is $5.45^2 = 29.7$

19 The new mean is $12 \times 2 + 4 = 28$

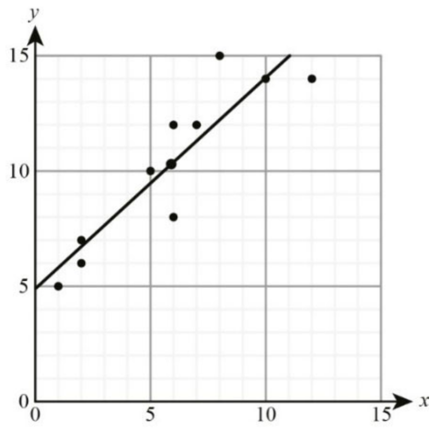
The new standard deviation is $10 \times 2 = 20$

20 Using the GDC, the lower quartile is 16 and the upper quartile is 28.5

21 a From the GDC: $r = 0.910$

b There is strong positive correlation between x and y.

22 a Something like:



b Approximately 7.5.

23 From GDC, $y = 0.916x + 4.89$

24 a i 13.1

ii 23.2

iii 5.58

b Only part **i**. Part **ii** is extrapolation and part **iii** is using a y-on-x line inappropriately.

Tip: A y-on-x regression line can only be used to predict values of y for given values of x; not the other way around.

25 a This is the expected number of text messages sent by a pupil who does not spend any time on social media in a day.

b For every additional hour spent on social media, the model predicts that the pupil will send 1.4 additional texts.

26 a Split the data into the first four points and the next five points and do a regression for each part separately.

$$L = \begin{cases} 4.19A - 0.259 & A < 6 \\ 0.830A + 25.4 & A > 6 \end{cases}$$

b Using the first part of the piecewise graph:

$L = 4.19 \times 3 - 0.259 = 12.3$

So expect a length of 12.3 cm.

27 $\frac{134}{200} = 0.67$

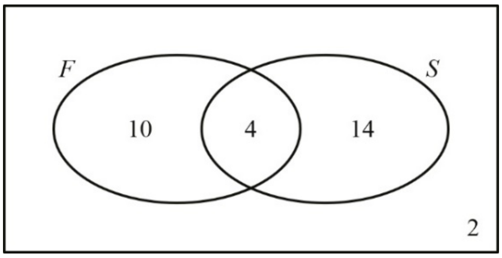
28 There are six possible outcomes of which three (2, 3 and 5) are prime, so the probability is $\frac{3}{6} = 0.5$

29 $\mathrm{P}(A') = 1 - \mathrm{P}(A) = 0.4$

30 $30 \times 0.05 = 1.5$

Tip: Remember that expected values should not be rounded to make them achievable.

31 We can illustrate this in a Venn diagram:



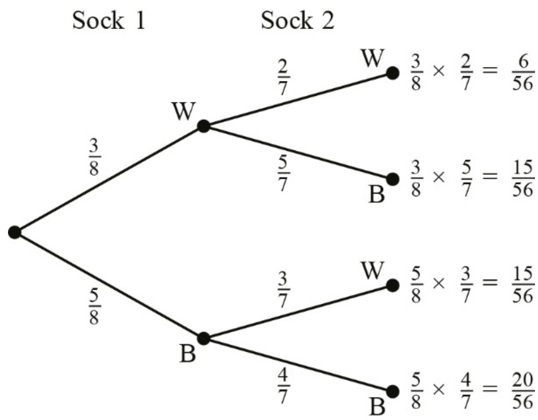
There are $14 - 4 = 10$ students who study only French.

There are $18 - 4 = 14$ students who study only Spanish.

Therefore, there are $10 + 14 + 4 = 28$ students who study either French or Spanish. This leaves 2 students who do not study either, so the probability is $\frac{2}{30} = \frac{1}{15}$

32 a There will be 7 socks left, of which 5 are black so it is $\frac{5}{7}$

b This is best illustrated using a tree diagram:



There are two branches relevant to the question.

White then black:

$$\frac{3}{8} \times \frac{5}{7} = \frac{15}{56}$$

Black then white:

$$\frac{5}{8} \times \frac{3}{7} = \frac{15}{56}$$

The total probability is $\frac{30}{56} = \frac{15}{28}$

33 a This can be best illustrated using a sample space diagram:

		1st roll			
		1	**2**	**3**	**4**
2nd roll	**1**	2	3	4	5
	2	3	4	5	6
	3	4	5	6	7
	4	5	6	7	8

There are 16 places in the sample space diagram. 6 of them have a score above 5 (shaded blue in the diagram) therefore the probability is $\frac{6}{16} = \frac{3}{8}$

b In the sample space diagram there are 6 scores above 5. Two of them are 7 so $\mathrm{P}(\text{score} = 7|\text{score} > 5) = \frac{2}{6} = \frac{1}{3}$

34 a $x = 100 - 40 - 30 - 20 = 10$

b There are 60 out of 100 students who prefer maths, so $\frac{60}{100} = \frac{3}{5}$

35 $\mathrm{P}(A \cup B) = 0.5 + 0.7 - 0.3 = 0.9$

36 $\mathrm{P}(A \cup B) = \mathrm{P}(A) + \mathrm{P}(B) - 0 = 0.6$

37 There are 70 people who prefer soccer. Out of these 40 prefer maths. So $P(\text{maths}|\text{soccer}) = \frac{40}{70} = \frac{4}{7}$

38 $\mathrm{P}(A \cap B) = \mathrm{P}(A)\mathrm{P}(B) = 0.24$

39 W can take three possible values: 0, 1 or 2.

$$\mathrm{P}(W = 0) = \mathrm{P}(BB) = \frac{4}{7} \times \frac{3}{6} = \frac{2}{7}$$

$$\mathrm{P}(W = 1) = \mathrm{P}(BW) + \mathrm{P}(WB) = \frac{4}{7} \times \frac{3}{6} + \frac{3}{7} \times \frac{4}{6} = \frac{4}{7}$$

$$\mathrm{P}(W = 2) = \mathrm{P}(WW) = \frac{3}{7} \times \frac{2}{6} = \frac{1}{7}$$

So

w	0	1	2
$\mathrm{P}(W = w)$	$\frac{2}{7}$	$\frac{4}{7}$	$\frac{1}{7}$

40 We can create a table:

$\boldsymbol{x}$	0	1	2
$\mathbf{P}(\boldsymbol{X}=\boldsymbol{x})$	k	$2k$	$3k$

The total probability is $k+2k+3k=6k$, which must equal 1 so $k=\frac{1}{6}$

41 $\mathrm{E}(X)=0.5\times 0.5+1\times 0.4+2.5\times 0.1=0.9$

42 a The probability of winning any prize is $0.095+0.005=0.1$

The probability of winning \$2000 is 0.005, so the conditional probability is $\frac{0.005}{0.1}=0.05$

b $\mathrm{E}(X)=0\times 0.9+10\times 0.095+2000\times 0.005=10.95$

$\mathrm{P}(X>10.95)=\mathrm{P}(X=2000)=0.005$

43 $\mathrm{E}(X)=-1\times 0.6+0\times 0.3+0.1k=0.1k-0.6$

If the game is fair then $\mathrm{E}(X)=0$ so:

$0.1k-0.6=0$

$0.1k=0.6$

$k=6$

44 The outcome of each trial is not independent of the previous trial.

45 Your calculator should have two functions – one which finds $\mathrm{P}(X=x)$, which we will use in part **a** and one which finds $\mathrm{P}(X\le x)$ which we will use in part **b**.

a $\mathrm{P}(X=2)=0.3456$

b To use the calculator, we need to write the given question into a cumulative probability:

$\mathrm{P}(X\ge 3)=1-\mathrm{P}(X\le 2)=1-0.683=0.317$

46 a If X is the number of heads then $\mathrm{E}(X)=np=10\times 0.6=6$

b $\mathrm{Var}(X)=10\times 0.6\times 0.4=2.4$ so the standard deviation is $\sqrt{2.4}\approx 1.55$

47 We know that about 68% of the data occurs within one standard deviation of the mean, but this gives negative values of time which are not possible.

48 a It is a symmetric, bell-shaped curve.

b The line of symmetry is approximately at 50, so this is a good estimate of the mean.

49 $P(11<X<15)$ can be found on the calculator – either directly or as $\mathrm{P}(X<15)-\mathrm{P}(X<11)$. It equals 0.625

50 Some calculators can deal with the given information directly, but some require you to first convert the information into a cumulative probability: $\mathrm{P}(X\le k)=0.3$. Using the inverse normal function on the calculator gives $k=92.1$

51 We first need to convert the original data into ranks. It does not matter whether the ranks are high to low or low to high. We will rank the lowest value as 1:

r_x	1	3	2	4
r_y	3	1	2	4

Then use the calculator to find the Pearson's product moment correlation coefficient of the ranks. The result is 0.2.

52 With tied ranks averaged, the ranks are:

r_x	1	2.5	2.5	4
r_y	2	2	4	2

Using the GDC to find the correlation coefficient of the ranks, the result is 0

53 r_s would be more appropriate as it measures any tendency to increase while r is only looking for linear relationships.

54 r_S, because it is less sensitive to outliers.

55 a $H_0: \mu = 0.4$; $H_1: \mu \neq 0.4$

b $H_0: \mu = 0.4$; $H_1: \mu > 0.4$

Tip: You could write these in words, but it is usually easier to use equations and inequalities where possible. Make sure that the null and alternative hypotheses are in terms of population parameters – in this case, the population mean (μ) rather than the sample mean ($\bar{x}$).

56

blond	brown	black
$100 \times \frac{1}{8} = 12.5$	$100 \times \frac{4}{8} = 50$	$100 \times \frac{3}{8} = 37.5$

Tip: Expected values should not be rounded to the nearest whole number.

57 First work out the probabilities of the binomial distribution, using your GDC:

Outcome	**0**	**1**	**2**
Probability	**0.36**	**0.48**	**0.16**

To find the expected frequencies, multiply each probability by 35:

Outcome	0	1	2
Expected frequency	12.6	16.8	5.6

58 The sample size is 90. The GDC can be used to find the probabilities; however, the ranges must be extended to all possible values otherwise the probabilities would not add up to 1:

x	$x < 120$	$120 \leq x < 130$	$130 \leq x < 140$	$140 \leq x$
Probability	0.309	0.383	0.242	0.0668
Expected frequency	27.8	34.5	21.8	6.01

59 a If you use the calculator to do a chi-squared test, the calculator works out the expected frequencies:

		Variable X		
		A	B	C
Variable Y	D	$\frac{176}{9}$	$\frac{156}{9}$	$\frac{136}{9}$
	E	$\frac{220}{9}$	$\frac{195}{9}$	$\frac{170}{9}$

b The appropriate test is a chi-squared test for independence. This can be done on your GDC. The output is:

$\chi^2 = 9.96$, two degrees of freedom. p-value is 6.86×10^{-3}

Since the p-value is less than 5%, there is significant evidence that the variables are dependent.

60 There are $10 + 15 + 12 + 13 = 50$ observations.

If all the outcomes are equally likely, the expected frequency of each one is $\frac{50}{4} = 12.5$

There are 4 groups of outcomes, so there are $4 - 1 = 3$ degrees of freedom.

All this can be put into the chi-squared goodness of fit test on the GDC to get:

$\chi^2 = 1.04$,

p-value $= 0.792$

Since $0.792 > 0.05$ there is not significance evidence that the outcomes are not equally likely.

Tip: Just because there is not significant evidence that the outcomes are not equally likely, does not mean that there is significant evidence that they are the same.

61 Since χ^2 is a measure of the distance between the observed and expected, a large value represents a big difference. The calculated χ^2 is larger than the critical value so there is significant evidence that the observed frequencies differ from the expected frequencies.

62 Using a one sample, one-tailed t-test from the GDC:

$\bar{x} = 12.2, s_{n-1} = 2.39, t = 2.06, p = 0.0542$

Since $p > 0.05$ there is no significant evidence that the mean is bigger than 10.

63 Using a one sample, two-tailed t-test from the calculator, $t = -3.46$ and p-value $= 5.29 \times 10^{-3}$.

The p-value is less than 5% therefore there is significant evidence that the mean length of newts is not 13 cm.

64 Using a two sample, two-tailed t-test with pooled variance (in this course you always used the pooled variance option):

$t = 0.248, p = 0.814$

Since the p-value is greater than 0.10, there is no significant evidence that the two groups have a different population mean.

65 Using a two sample, one-tailed test with pooled variance:

$t = 2.27, p = 0.0428$

Since the p-value is less than 5%, there is significant evidence that group A is drawn from a population with a larger mean than group B.

66 It assumes that the data is drawn from a normal distribution.

(There are lots of other assumptions, including that the sampling is random and representative and that in the two-sample case the two groups have equal population variance; however, it is the normal distribution one which is focused on in this course.)

67 a An internet survey is unlikely to be valid as it will not be representative of the entire community. For example, not everyone has internet access or small groups might post repeatedly.

b This question is loaded – it is pushing towards the answer YES. People who might agree with one part but not the other have no valid option.

68 The number of doctors would vary depending on the population size. Number of doctors per thousand people might be a better measure.

Tip: In these type of questions there is no single right answer. Nor are you expected to have any specialist knowledge. Anything that shows suitable engagement would be given credit – for example, number of doctors does not necessarily imply good health outcomes, particularly in unequal societies, so you could also have suggested looking at life expectancy.

69 Entry B contains a year of birth which is not feasible for a high school student in 2021.

Entry C contains a month which is not possible. It might be that the month and year entries have been switched.

70 The expected frequencies must be greater than five. This can be achieved by grouping the last two groups:

Values	$0 \le x < 5$	$5 \le x < 10$	$10 \le x < 20$
Observed frequency	4	7	14
Expected frequency	6.1	7.5	11.4

71 The average number of successes is $\frac{0\times31+1\times48+2\times54+3\times46+4\times21}{31+48+54+46+21} = 1.89$

The best estimate of p is $\frac{1.89}{4} = 0.4725$

Using the GDC, expected frequencies are therefore (to 2 d.p.)

Successes	0	1	2	3	4
Probability	0.077427	0.277415	0.372735	0.222581	0.049843
Expected frequency ($p \times 200$)	15.49	55.48	74.55	44.52	9.97

Check that all the expected frequencies are over 5, then we can use the GDC to find χ^2, which equals 34.47. There are 5 groups so there are $5 - 1 - 1 = 3$ degrees of freedom. (subtracting 1 because the total is constrained to be 20 and 1 because the probability is constrained to be 0.4725)

The p-value is $1.57 \times 10^{-7} < 0.05$ therefore there is significant evidence that the data is drawn from a population which is not normally distributed.

72 a Conclusions are reliable if similar conclusions would be reached on each occasion the test is conducted in similar circumstances.

b A process is valid if it is measures what you really want to measure.

73 The teacher could give another similar test at a later date to see if the students were still getting similar results.

74 She could use an expert panel to judge whether the questions assess appropriate ways of measuring motivation or look at the same students at a later date to see whether those who she predicted were motivated showed the signs of actually being motivated.

75 From the GDC: $y = 0.423x^2 - 1.80x + 0.0947$

76 $y = 1.78x^3 + 2.63x^2 - 4.13x + 5.23$

77 $y = 1.99e^{0.498x}$

78 $y = 1.90x^{4.03}$

79 From the calculator:

Therefore, the regression line is $y = 2.00\sin(0.999x + 0.204) + 3.00$

80 The power regression is a better fit since it has a lower SS_{res}.

SS_{res} depends on how many data points there are, so is not normally a good way to compare two different data sets.

81 For the exponential regression model, from the GDC $R^2 = 0.9995$. For the power regression model $R^2 = 0.9997$. Both show an excellent fit, as both are very close to 1. The power model is a slightly better fit since it has a larger R^2 value.

The sine regression curve has more parameters, so it would be expected to have a better fit, therefore R^2 does not provide a fair comparison with the power regression curve.

82 For the linear model $R^2 = (-0.72)^2 = 0.5184$. This is less than 0.62 so the exponential regression model has a better fit.

83 $\mathrm{E}(3X - 5) = 3\,\mathrm{E}(X) - 5 = 1$

84 $\mathrm{Var}(1 - 2X) = (-2)^2\,\mathrm{Var}\,X = 28$

85 $\mathrm{E}(2X - Y + 1) = 2\,\mathrm{E}(X) - \mathrm{E}(Y) + 1 = 9$

86 $\mathrm{Var}\left(\frac{X_1+X_2}{2}\right) = \left(\frac{1}{2}\right)^2 (\mathrm{Var}(X_1) + \mathrm{Var}(X_2)) = \frac{1}{4}(\mathrm{Var}(X) + \mathrm{Var}(X)) = \frac{3}{2}$

87 $\bar{x} = \frac{3+5+7+8+10}{5} = 6.6$, so this is an unbiased estimate of μ.

88 From the calculator, $s_{n-1} = 2.701851$ so $s_{n-1}^2 = 7.3$

89 $s_{n-1}^2 = \frac{n}{n-1} s_n^2 = \frac{4}{3} \times 12^2 = 192$

90 If A is the mass of an apple and P is the mass of a pear, then $A \sim \mathrm{N}\,(100,100)$ and $\sim \mathrm{N}\,(160,400)$. We are interested in the random variable $P - 2A$. The mean of this is $160 - 2 \times 100 = -40$ and the variance is $400 + (-2)^2 \times 100 = 800$ so $P - 2A \sim \mathrm{N}(-40,800)$.

We need $P(P - 2A > 0) \approx 0.0786$ from the GDC.

91 $\bar{X} \sim \mathrm{N}\left(40, \frac{25}{10}\right)$ so from the GDC $P(\bar{X} < 38) \approx 0.103$

92 Using the CLT, $\bar{X} \sim \mathrm{N}\left(12, \frac{20}{50}\right)$ so from the GDC $P(\bar{X} > 11) \approx 0.943$

93 From the GDC:

$8.69 < \mu < 15.7$

94 From the GDC:

$14.6 < \mu < 25.1$

95 No, because $\mu = 0$ is contained within the confidence interval.

96 The average rate at which worms are found is unlikely to be constant as it will vary depending upon the environment.

97 $\mathrm{E}(X) = 2.5$ so $\mathrm{Var}(X) = 2.5$

$\mathrm{E}(2X) = 2\,\mathrm{E}(X) = 5,$

$\mathrm{Var}(2X) = 2^2\,\mathrm{Var}(X) = 10$

Since $\mathrm{Var}(2X) \neq \mathrm{E}\,(2X)$ this cannot follow a Poisson distribution.

98 From the GDC, $P(X \leq 3) \approx 0.625$

99 The total number of insects, T, follows a Poisson distribution with mean 6.4.

$P(T > 8) = 1 - P(T \leq 8) \approx 0.197$

100 Since the test statistic lies within the critical region, the null hypothesis should not be rejected.

101 This is a one-tailed test. Since the population standard deviation is given, a z-test is appropriate. Using a GDC, the p-value is $7.83 \times 10^{-4} < 0.05$; therefore, there is significant evidence that the population mean is below 20.

102 The differences in the assessments are:

Student	Andi	Beth	Claude	Deepti
Difference	4	5	-2	1

Since no population variance is given, a t-test is appropriate. We are looking for any difference, so a two-tailed test is used with the null hypothesis being that the mean difference is 0. From the GDC, $t = 1.26$, $p = 0.295 > 0.05$ therefore there is no significant evidence of a difference between the two tests.

103 First, find the critical value of Z. Since the test is two tailed, half of the type I error occurs in either tail, so the upper boundary has 2.5% above it, therefore 97.5% below it. The inverse normal of 0.975 is 1.96. Since the region is symmetric about 0 the critical region is $Z < -1.96$ or $Z > 1.96$. We can substitute in $Z = \frac{\bar{X}-\mu}{\sigma/\sqrt{n}} = \frac{\bar{X}-100}{20/\sqrt{16}} = \frac{\bar{X}}{5} - 20$.

Therefore,

$\frac{\bar{X}}{5} - 20 < -1.96$ or $\frac{\bar{X}}{5} - 20 > 1.96$

$\frac{\bar{X}}{5} < 18.04$ or $\frac{\bar{X}}{5} > 21.96$

$\bar{X} < 90.2$ or $\bar{X} > 110$

104 If X is the number of 1's rolled, then under the null hypothesis, $X \sim B\left(100, \frac{1}{6}\right)$.

The p value is $P(X \geq 21) = 1 - P(X \leq 20) = 0.152 > 0.05$ so there is no significant evidence that the die rolls 1 more often than expected.

105 Finding the p value for the different possible outcomes, using the same method as in question 106:

X	p –value
21	0.151888
22	0.100183
23	0.063050
24	0.037864
25	0.021703
26	0.011877

The first value of X which has a p-value below 0.05 is 24, so the critical value is $X = 24$ and the critical region is $X \geq 24$

Tip: Some calculators are better than others at automating the process of creating the table here – explore whether your calculator can do this.

106 Let X be the number so lions in a 5 km^2 area. Then $X \sim Po(5 \times 4.1)$.

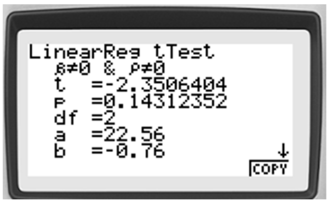
The p-value is $P(X \leq 14) = 0.0869 > 0.05$ therefore there is no significant evidence that the number of lions has decreased.

107 Finding the p value for the different possible outcomes, using the same method as in question 108:

X	$p-$ value
14	0.086904
13	0.053706
12	0.031034

The first value of X which has a p-value below 0.05 is 12, so the critical value is $X = 1$ 2and the critical region is $X \leq 12$

108 From the calculator:



The p-value is $0.143 > 0.1$ therefore there is not significant evident that the population correlation coefficient differs from zero.

109 a The null hypothesis is that the accused person is innocent.

b A type I error is finding an innocent person guility.

c A type II error is finding a guility person innocent.

110 a This is a Z-test. Since there are two parts to the critical region it is two tailed and it is symmetric about the null hypothesis value, since $\frac{66.84+93.16}{2} = 80$ the null hypothesis is $\mu = 80$ and the alternative hypothesis is $\mu \neq 80$.

b For a continuous variable the significance level is the same as the probability of a type I error. If the null hypothesis is true then $\bar{X} \sim \mathrm{N}\left(80, \frac{40^2}{25}\right)$. A type I error is the null hypothesis being rejected when it was true, so this happens in this circumstance if $\bar{X} > 93.16$ or $\bar{X} < 66.84$. Using the GDC normal distribution calculator, this is $P(\bar{X} > 93.16) + P(\bar{X} < 66.84) = 0.09997$, i.e. a 10% significance level.

111 If $\mu = 90$ then $\bar{X} \sim \mathrm{N}\left(90, \frac{40^2}{25}\right)$. Whenever we fail to reject the null hypothesis we are making a type II error, and this occurs if $66.84 < \bar{X} < 93.16$. From the GDC, $P(66.84 < \bar{X} < 93.16) \approx 0.652$

112 If the null hypothesis is true then $X \sim B(10,0.5)$. A type I error is rejecting the null hypothesis if this is the case. The probability of this occurring is $P(X \geq 9) = 1 - P(X \leq 8) \approx 0.0107$

113 With the new information we know that $X \sim B(10,0.6)$. A type II error is now not rejecting the null hypothesis. The probability of this occurring is $P(X \leq 8) \approx 0.954$

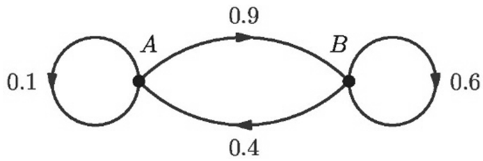
114 It can be wet after being wet the previous day (with probability 0.7) or dry the previous day (with probability 0.4).

$W_{n+1} = 0.7W_n + 0.4D_n$

$D_{n+1} = 0.3W_n + 0.6D_n$

$\boldsymbol{T} = \begin{pmatrix} 0.7 & 0.4 \\ 0.3 & 0.6 \end{pmatrix}$

115



116 Using technology, the state is

$\begin{pmatrix} 0.5 & 0.75 \\ 0.5 & 0.25 \end{pmatrix}^4 \begin{pmatrix} 0.8 \\ 0.2 \end{pmatrix} \approx \begin{pmatrix} 0.60 \\ 0.40 \end{pmatrix}$

117 $\boldsymbol{A}$ cannot be a Markov chain because the columns do not add up to 1.

$\boldsymbol{B}^2 = \begin{pmatrix} 0.35 & 0.425 & 0.32 \\ 0.1 & 0.1125 & 0.17 \\ 0.55 & 0.4625 & 0.51 \end{pmatrix}$. So, all subsequent powers of $\boldsymbol{B}$ will contain only positive entries so this is a regular Markov chain.

$\boldsymbol{C}$ does not produce a regular Markov chain because $\boldsymbol{C}^n = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0.5 & 0.5 \\ 0 & 0.5 & 0.5 \end{pmatrix}$ which still contains zeros.

118 At the equilibrium value:

$\begin{pmatrix} 0.6 & 0.9 \\ 0.4 & 0.1 \end{pmatrix} \begin{pmatrix} O \\ C \end{pmatrix} = \begin{pmatrix} O \\ C \end{pmatrix}$

This becomes the equation:

$-0.4O + 0.9C = 0$

We also know that

$O + C = 1$

Solving these simultaneously on the calculator gives

$O = \frac{9}{13}, \ C = \frac{4}{13}$

So, the long-term probability that the plant is open is $\frac{9}{13} \approx 0.692$

5 Calculus

1

x	$\frac{\sin(3x)}{0.2x}$
10	0.25
5	0.2588
1	0.2617
0.1	0.2618

The limit is 0.26

2 Look at the values on the graph close to $x = 2$

The limit is 0.5

3 The derivative is $\frac{dy}{dx} = 12 - 5 = 7$

4 'Rate' means $\frac{dA}{dt}$; 'decreases' means that the rate of change is negative.

$$\frac{dA}{dt} = -kA$$

5 The y value is $f(x)$ and the gradient is $f'(x)$. So when $y = 4$, $f'(x) = -1$.

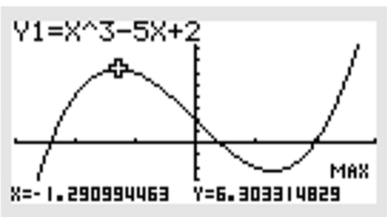
6 Use $\Delta x = x_Q - 4, \Delta y = y_Q - 2$ and gradient $= \frac{\Delta y}{\Delta x}$

x_Q	y_Q	Δx	Δy	Gradient of PQ
5	2.236	1	0.236	0.236
4.1	2.025	0.1	0.025	0.248
4.01	2.002	0.01	0.002	0.250
4.001	2.000	0.001	0.000	0.250

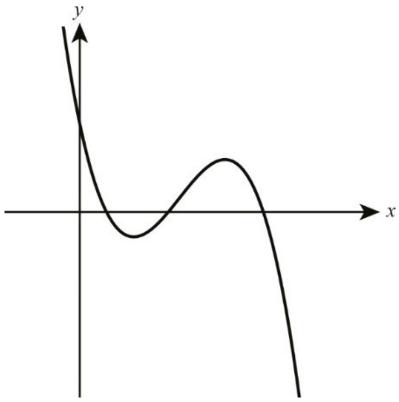
The gradient is ≈ 0.25

7 $f'(x)$ is where the graph is decreasing, which is between the two turning points.

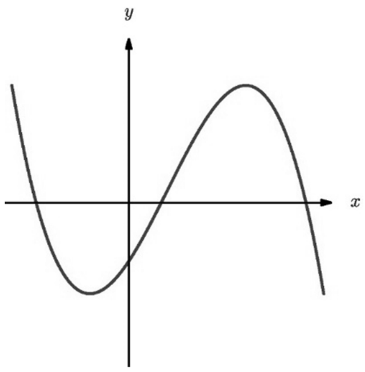
$-1.29 < x < 1.29$



8 The gradient starts positive but decreasing, then changes to negative, then back to positive and then to negative again.



9 The gradient starts off negative, so $f(x)$ is decreasing. It then increases, and then decreases again.



10 $\frac{dy}{dx} = 8x + \frac{1}{2}x^{-6} - 3$

11 a $f(x) = 12x^2 - 3x^6$ so $f'(x) = 24x - 18x^5$

b $f(x) = 1 - \frac{3}{2}x^{-4}$, so $f'(x) = \frac{6}{x^5}$ [or $6x^{-5}$]

c $f(x) = \frac{4}{5}x - \frac{3}{5} + \frac{1}{5}x^{-1}$, so $f'(x) = \frac{4}{5} - \frac{1}{5x^2}$ [or $\frac{4}{5} - \frac{1}{5}x^{-2}$]

12 $f'(x) = 8x + 2x^{-2}$

$$f'(2) = 16 + \frac{2}{4} = 16.5$$

13 $\frac{dy}{dx} = 12 - 5x^{-2}$

$12 - 5x^{-2} = 2$

$$\frac{5}{x^2} = 10, x^2 = \frac{1}{2}$$

$$x = \pm\frac{1}{\sqrt{2}}$$

14 $\frac{dy}{dx} = 2x = 8, y = 16 - 3 = 13$

Tangent: $y - 13 = 8(x - 4)$

$y = 8x - 19$

15 $\frac{\mathrm{d}y}{\mathrm{d}x} = 3 + 2x^{-2} = 3 + \frac{2}{4} = \frac{7}{2}$

$y = 6 - \frac{2}{2} = 5$

Normal: $y - 5 = -\frac{2}{7}(x - 2)$

$$y = -\frac{2}{7}x + \frac{39}{7}$$

16 $\frac{\mathrm{d}y}{\mathrm{d}x} = 2x$, so the tangent at $(a, a^2 - 3)$ is:

$y - (a^2 - 3) = 2a(x - a)$

When $x = 0, y = -12$:

$-9 - a^2 = -2a^2$

$a = \pm 3$

17 Use GDC to find the gradient and to draw the tangent.

a gradient $= -0.021$

b $y = -0.021x + 0.13$

18 Use GDC to sketch the graph of $\frac{\mathrm{d}y}{\mathrm{d}x}$ and intersect it with $y = 2$. The coordinates are $(0.5, -0.098)$.

19 $3x^3 - 3x^{-2} + c$

20 $\int \frac{1}{2}x^3 - \frac{3}{2}x^{-2}\mathrm{d}x = \frac{1}{8}x^4 + \frac{3}{2}x^{-1} + c$

21 Use GDC: $\int_2^3 2x^3 - 1\,\mathrm{d}x = 31.5$

22 Integrate: $y = \int 4x + 2\,\mathrm{d}x = 2x^2 + 2x + c$

Use $y = 3, x = 2: 3 = 2(2^2) + 2(2) + c \Rightarrow c = -9$

So $y = 2x^2 + 2x - 9$

23 $\frac{\mathrm{d}y}{\mathrm{d}x} = 6x^2 - 2ax = 0$

$2x(3x - a) = 0$

$x = 0$ or $\frac{a}{3}$

24 Sketch $y = \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{2}{x} + \sqrt{x}\right)$ and find the intersection with the x-axis.

$x = 2.52$

25 From the graph: $(-0.281, 2.12)$.

26 Use the graph, checking the local minimum $(3.46, -33.3)$ and the end point $(-5, -62.5)$. The smallest value is -62.5

27 $S = x^2 + 4 \times \left(x \times \frac{32}{x^2}\right) = x^2 + \frac{128}{x}$

Draw the graph for $x > 0$ to find that the minimum value is 48 cm^2.

28 Using the trapezoidal rule with $h = 0.5$:

$$\int_2^5 \mathrm{f}(x)\mathrm{d}x \approx 0.25[1.6 + 0.8 + 2(2.1 + \cdots + 1.5)] = 5.65$$

29 Use GDC to create a table, with x-values spaced by $\frac{2}{5} = 0.4$.

x	0	0.4	0.8	1.2	1.6	2
y	0	0.3031	0.4595	0.5223	0.5278	0.5

Trapezoidal rule with $h = 0.4$÷

$$\text{Area} \approx 0.2\{0 + 0.5 + 2(0.3031 + 0.4595 + 0.5223 + 0.5278)\} = 0.825$$

30 $\frac{\mathrm{d}y}{\mathrm{d}x} = 3\cos x + 5\sin x$

31 a Using the chain rule with $y = u^{\frac{1}{2}}$ and $u = 3x^2 - 1$:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{2}(3x^2 - 1)^{-\frac{1}{2}}(6x) = \frac{3x}{\sqrt{3x^2 - 1}}$$

b Using the chain rule with $y = 2u^3$ and $u = \sin(5x)$, and remembering that $\sin(5x)$ differentiates to $5\cos(5x)$:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 6\sin^2(5x) \times 5\cos(5x) = 30\sin^2(5x)\cos(5x)$$

32 $4\mathrm{e}^{-3x} - 12x\mathrm{e}^{-3x}$

33 Using the quotient rule with $u = \ln x$ and $v = 4x$:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{4x\frac{1}{x} - 4\ln x}{16x^2} = \frac{4 - 4\ln x}{16x^2} = \frac{1 - \ln x}{4x^2}$$

34 $\frac{1}{8\pi}$ cm s^{-1}

35 $6x + \frac{3}{x^2}$

36 $\mathrm{f}'(x) = \cos x + \sin x$, $\mathrm{f}'\left(\frac{3\pi}{4}\right) = -\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} = 0$

$$\mathrm{f}''(x) = -\sin x + \cos x, \mathrm{f}''\left(\frac{3\pi}{4}\right) = -\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2} < 0$$

$$y = \mathrm{f}\left(\frac{3\pi}{4}\right) = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} = \sqrt{2}$$

37 Concave up means $\mathrm{f}''(x) > 0$. So, $\mathrm{f}''(x) = 30x - 4 > 0 \therefore x > \frac{2}{15}$

38 Concave-down means that the graph curves downwards, which is at the points B, D and E.

39 $\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 60x^3 - 120x^2 = 60x^2(x-2)$

So, $\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 0$ when $x = 0$ or 2.

Check for change in sign of $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}$:

x	-1	1	3
$\frac{\mathrm{d}^2y}{\mathrm{d}x^2}$	< 0	< 0	> 0

The only point of inflection is at $x = 2$.

40 $\left(2 \div \frac{1}{3}\right)x^{\frac{1}{3}} + \frac{4}{3}\ln|x| + c = 6x^{\frac{1}{3}} + \frac{4}{3}\ln|x| + c$

41 $\left[-\frac{1}{2}\cos(2x)\right]_0^{\frac{\pi}{6}} = -\frac{1}{2}\left(\frac{1}{2} - 1\right) = \frac{1}{4}$

42 $\frac{2\mathrm{e}^{4x}}{4} + \frac{3\mathrm{e}^{-\frac{1}{3}x}}{-\frac{1}{3}} = \frac{1}{2}\mathrm{e}^{4x} - 9\mathrm{e}^{-\frac{1}{3}x} + c$

43 a $4 \times \frac{\sin^3 x}{3} = \frac{4}{3}\sin^3 x + c$

b $\frac{1}{2}\int\left(\frac{2x}{x^2+3}\right)\mathrm{d}x = \frac{1}{2}\ln(x^2+3) + c$

44 a -15.75

b 21.1

45 $\int_2^6 \mathrm{e}^{\frac{y}{2}}\,\mathrm{d}y = \frac{1}{2}(\mathrm{e}^3 - \mathrm{e})$

46 $\int_0^{\pi} \pi \sin^2 x\ \mathrm{d}x = 4.93$ (from GDC)

47 $\int_1^9 \pi\left(\sqrt{y}\right)^2\mathrm{d}y = \left[\pi\frac{y^2}{2}\right]_1^9 = 40\pi$

48 $v = \frac{\mathrm{d}s}{\mathrm{d}t} = 15\cos(5t)\,, a = \frac{\mathrm{d}v}{\mathrm{d}t} = -75\sin(5t)$

When $t = 2, s = 40.8\ \mathrm{m\ s^{-2}}$

(or find $\frac{\mathrm{d}^2s}{\mathrm{d}t^2}$ at $t = 2$ using GDC)

49 Velocity: $v = \frac{\mathrm{d}s}{\mathrm{d}t} = -0.6\mathrm{e}^{-0.2t} = -0.270$

(or find $\frac{\mathrm{d}s}{\mathrm{d}t}$ using GDC)

So, speed = 0.270 m s^{-1}

50 $\dot{x}$ is the velocity, so the acceleration is $\ddot{x} = 6t - 2$

51 Using $a = v\frac{\mathrm{d}v}{\mathrm{d}x}$:

$a = (4\sin x)(4\cos x)$

$= 16\sin 2\cos 2 = -6.05$

52 $4 + \int_2^5 \frac{1}{\sqrt{t+3}}\mathrm{d}t = 5.18$ m

53 13.8 m

54 The rate of change is $\frac{dA}{dt}$. 'Decrease' means that the rate is negative.

$$\frac{dA}{dt} = -k\sqrt[3]{A}, \quad \text{where } k \text{ is a positive contstant}$$

55 $\frac{dy}{dx} = x(y^2 + 1)$

$$\Rightarrow \int \frac{dy}{y^2 + 1} = \int x \, dx$$

$$\Rightarrow \arctan y = \frac{1}{2}x^2 + c$$

$$y = \tan\left(\frac{1}{2}x^2 + c\right)$$

56 Separate variables and integrate:

$$\int \frac{1}{y^2} \, dy = \int 4x \, dx$$

$$-\frac{1}{y} = 2x^2 + c$$

The constant is a part of the general solution.

$$y = -\frac{1}{2x^2 + c}$$

57 $\int \frac{dy}{y+2} = \int (x-1)dx$

$$\ln(y + 2) = \frac{1}{2}(x-1)^2 + c$$

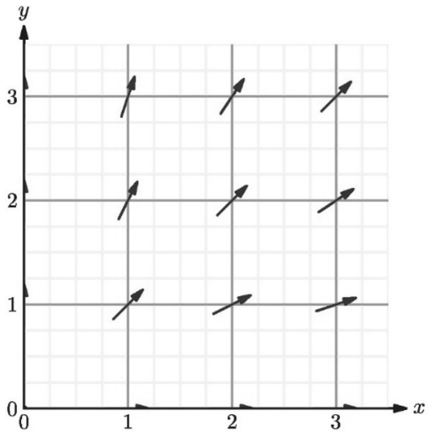
Using $y = 1, x = 1$: $c = \ln 3$

$$y + 2 = e^{\frac{1}{2}(x-1)^2 + \ln 3}$$

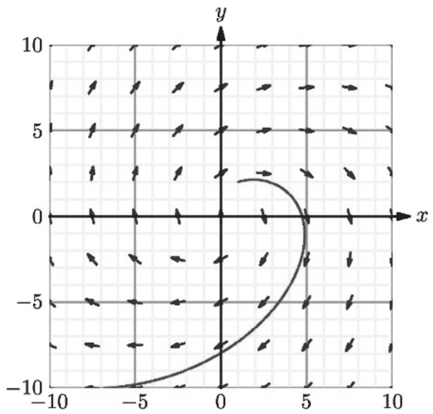
$$y = 3e^{\frac{1}{2}(x-1)^2} - 2$$

58 Calculate the gradient at each point first:

y / **x**	**1**	**2**	**3**
1	1	2	3
2	$\frac{1}{2}$	1	$\frac{3}{2}$
3	$\frac{1}{3}$	$\frac{2}{3}$	1



59 Start from $(1, 2)$ and follow the slope lines.



60

x	y
0	2.0000
0.1	2.0909
0.2	2.1723
0.3	2.2419
0.4	2.2983

So, $y = 2.298$

61 Make a table showing values of t, x and y at each step.

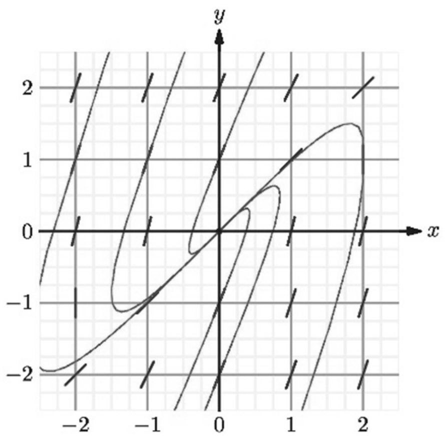
t	x	y
0.0	0.00	2.00
0.1	0.20	2.40
0.2	0.44	2.90
0.3	0.75	3.52
0.4	1.15	4.30
0.5	1.67	5.28

62 The matrix is

$$A = \begin{pmatrix} 1 & -3.5 \\ 14 & 1 \end{pmatrix}$$

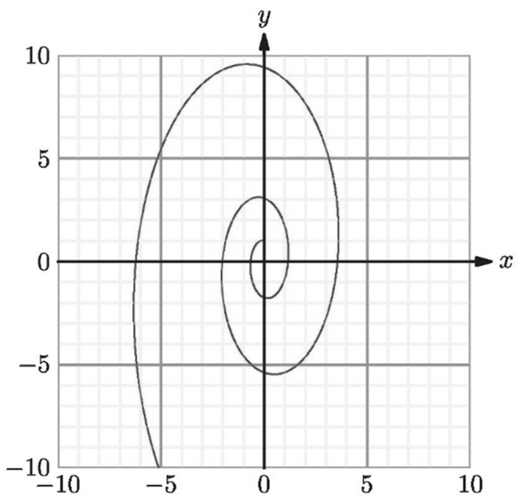
Its eigenvalues are $1 + 7\text{i}$ and $1 - 7\text{i}$.

63 The eigenvalues of the matrix $\begin{pmatrix} 1 & -2 \\ 4 & -5 \end{pmatrix}$ are -1 and -3, which are real and negative, so the trajectories move towards the origin.



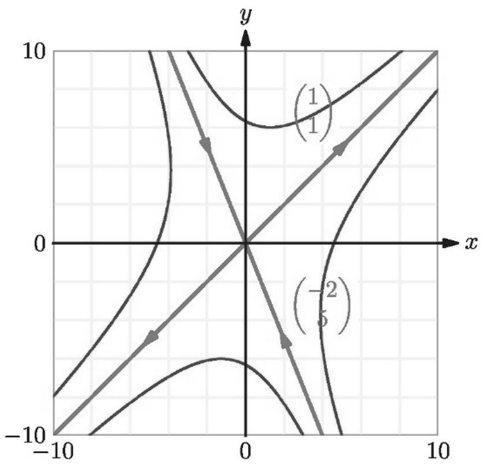
64 From Question 62, the eigenvalues are $1 + 7\text{i}$ and $1 - 7\text{i}$. The real part is positive, so the trajectories spiral away from the origin.

When $x = 0$, $\frac{\text{d}x}{\text{d}t} = -3.5y$, so x decreases when y is positive (e.g. when $x = 0$ and $y = 1$, the arrow will be horizontal and pointing to the left). This means that the trajectories spiral anticlockwise.



65 The eigenvalues of the matrix $\begin{pmatrix} 2 & 2 \\ 5 & -1 \end{pmatrix}$ are −3 and 4. Since one is positive and one negative, the origin is a saddle point.

The eigenvectors are $\begin{pmatrix} 2 \\ -5 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$, so the trajectories approach the origin long the direction $\begin{pmatrix} 2 \\ -5 \end{pmatrix}$ and move away from it along the direction $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$.



66 The equilibrium points are $(0, 2)$, which is stable, and $(2, 1)$, which is a saddle.

67 For the matrix $\begin{pmatrix} 2 & 2 \\ 5 & -1 \end{pmatrix}$, the eigenvalues are $-3, 4$ and the eigenvectors are $\begin{pmatrix} 2 \\ -5 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

Hence the general solution is

$$\begin{pmatrix} x \\ y \end{pmatrix} = A\mathrm{e}^{-3t}\begin{pmatrix} 2 \\ -5 \end{pmatrix} + B\mathrm{e}^{4t}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

68 Re-write the equation as a system:

$$\begin{cases} \dfrac{\mathrm{d}x}{\mathrm{d}t} = y \\ \dfrac{\mathrm{d}y}{\mathrm{d}t} = 2xy \end{cases}$$

Make a table of values:

t	*x*	*y*
0.00	1.00	2.00
0.05	1.10	2.20
0.10	1.21	2.44
0.15	1.33	2.74
0.20	1.47	3.10

So, $x \approx 1.47$

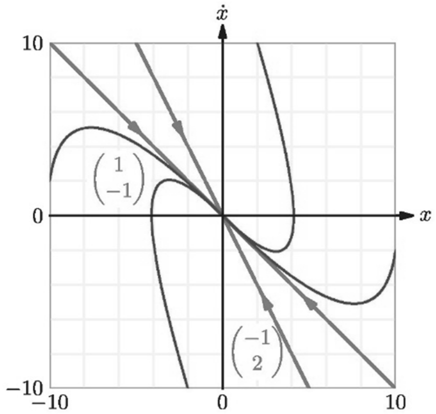
69 Write $y = \frac{\mathrm{d}x}{\mathrm{d}t}$, then $\frac{\mathrm{d}y}{\mathrm{d}t} = -2x - 3y$

The corresponding matrix is

$\begin{pmatrix} 0 & 1 \\ -2 & -3 \end{pmatrix}$

With eigenvalues $-1, -2$.

The eigenvalues are both negative, so the solutions curves move towards the origin.



70 From Question 69, the eigenvalues are $-1, -2$ and the corresponding eigenvectors $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ -2 \end{pmatrix}$.

So, $\begin{pmatrix} x \\ \dot{x} \end{pmatrix} = A\mathrm{e}^{-t}\begin{pmatrix} 1 \\ -1 \end{pmatrix} + B\mathrm{e}^{-2t}\begin{pmatrix} 1 \\ -2 \end{pmatrix}$

Hence the general solution is

$x = A\mathrm{e}^{-t} + B\mathrm{e}^{-2t}$